

OPEN BOOK FOLIATION

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ABSTRACT. We define open book foliation and study its properties and applications to contact geometry.

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1. INTRODUCTION

In his seminal work [1], Bennequin showed that there is an “exotic” contact structure ξ_{ot} on S^3 , which is homotopic to the standard contact structure ξ_{std} as a 2-plane field but it

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is *not* contactomorphic to ξ_{std} . In contemporary notions, what Bennequin showed is that ξ_{std} is tight, whereas ξ_{ot} is overtwisted. In order to distinguish these contact structures, he studied closed braids and certain foliations on their Seifert surfaces induced by the contact structures. Since then Bennequin's method has been developed into two directions.

One direction is the theory of characteristic foliation and convex surfaces. Eliashberg used the characteristic foliations to show Bennequin-Eliashberg inequality for tight contact 3-manifolds [17] and generalized Bennequin's result for the tight contact 3-sphere. The characteristic foliation also plays an important role in his classification of overtwisted contact structures [16]. In [25] Giroux developed the notion of characteristic foliation into convex surface theory, which provides us cut-and-paste technique to study contact structures and leads to classification of the tight contact structures for various 3-manifolds. See [29] for basic techniques in convex surface theory.

The other direction is the theory of braid foliation developed in a series of papers by Birman and Menasco [5, 6, 7, 8, 9, 10, 11]. One of its highest achievements is "Markov theorem without stabilization" which states that given two closed braid representatives of any link in \mathbb{R}^3 are transformed to each other in very controlled manner [11]. Moreover Birman-Menasco applied it to contact geometry and constructed examples of transversely non-simple knots in the standard contact 3-sphere: knots having the same topological type and the same self-linking number but being not transverse isotopic [12]. Their examples are closed 3-braids related by negative flypes. An analysis of Markov tower, a sequence of braid moves which relates one to the other, tells that the two closed braids represent distinct transverse knots. See also [3] for basic techniques.

In this paper we introduce *open book foliation* to integrate the above two methods, which appear to have origins in Bennequin's work. Our starting point is Giroux correspondence [27]: there is a one-to-one correspondence between the contact 3-manifolds up to contact isotopy and the open book decompositions of 3-manifolds up to positive stabilizations. An open book decomposition naturally induces a singular foliation on an embedded surface that reflects the compatible contact structure. The open book foliation is the induced foliation satisfying some additional conditions. As we see in Theorem 3.5, the requirements are not so restrictive that any embedded surface can be isotoped to admit an open book foliation.

Although the two foliations share numerous common properties (cf. Theorem 4.1), the open book foliation is more rigid than the characteristic foliation and it has stronger combinatorial nature. For instance, compared to the characteristic foliation, it is easier to obtain the whole picture of the open book foliation on a given surface. The open book foliation is a particularly convenient tool to study overtwisted contact structures. It helps us to see or construct an overtwisted disc explicitly, rather than just to show overtwistedness indirectly.

Once we develop simple basic notions and machineries of the open book foliation, we deduce various old and new results:

As a new result, we prove in Theorem 5.4 a self-linking number formula for closed braids in arbitrary open books, which generalizes Bennequin's formula [1] and subsequent works

in [36, 37]. Interestingly, our formula suggests that there is an unexpected relationship between contact structures and the Johnson-Morita homomorphisms in mapping class group theory.

In Section 6, we use the open book foliation to give alternative and combinatorial proofs to two fundamental results in contact geometry: Bennequin-Eliashberg inequality [17], and Honda-Kazez-Matić's characterization of monodromy for overtwisted contact structures [30].

We close the paper by giving an infinite family of open books that are right-veering and non-destabilizable but compatible with overtwisted contact structures, which answers negatively to a question of Honda-Kazez-Matić [30]. Our family generalizes the previously known examples by Lekili [38] and Lisca [39], but our proof of overtwistedness is more direct.

2. ORIGINS OF OPEN BOOK FOLIATION

In this section we briefly review the braid foliation and the characteristic foliation. They motivate our study of the open book foliation. We extend the braid foliation to the open book foliation. While, many applications of the open book foliations are derived from problems in characteristic foliation theory.

2.1. Braid foliation. In Birman-Menasco's braid foliation theory [5, 6, 7, 8, 9, 10, 11], braids are geometric objects: Let A be an oriented unknot in S^3 . As is well-known, A is a fibered knot: The fibration $\pi : S^3 \setminus A \rightarrow S^1$ is rather easy to see since $S^3 \setminus A$ is homeomorphic to the solid torus $D^2 \times S^1$, so π is nothing but the trivial projection $D^2 \times S^1 \rightarrow S^1$. Typically, we regard $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and identify A with a union of z -axis and the point at infinity ∞ . With the cylindrical coordinates (r, θ, z) of \mathbb{R}^3 , the fibration is given by the projection $(r, \theta, z) \mapsto \theta$. An oriented link $L \subset S^3$ is called a *closed braid* with respect to A (and π) if L is disjoint from A and positively transverse to each fiber $S_\theta = \pi^{-1}(\theta)$. In other words, L winds around the z -axis in the positive direction.

Now let us consider an incompressible Seifert surface F of L , or an essential closed surface $F \subset S^3 \setminus L$. The intersection of F and the fibers $\{S_\theta \mid \theta \in S^1\}$ induces a singular foliation \mathcal{F} on F . By putting F in a “nice” position, \mathcal{F} satisfies certain properties, which are generalized below in Section 3 into properties **(OF i)**–**(OF iv)** of the open book foliation. This \mathcal{F} is called the *braid foliation*.

The foliation \mathcal{F} encodes both geometric and algebraic features of the closed braid L . Here is a simple example: Let L be the closure of the braid word σ_1 in Artin braid group B_2 , and let F be its Bennequin surface that consists of two discs and one positively twisted band. See Figure 1-(a), where F and its braid foliation \mathcal{F} are depicted (the meaning of symbol \oplus will be made clear in Section 3.1.1). Now we collapse the twisted band and a disc to simplify \mathcal{F} and get the trivial braid as in Sketch (b). Algebraically, this corresponds to *destabilization* of σ_1 . Thus, the braid foliation \mathcal{F} detects that L admits destabilization.

In general, if \mathcal{F} can be “simplified” then L is also “simplified”. Moreover, as the above example suggests, a simplification of \mathcal{F} can be understood as a certain operation on closed braids. Therefore by modifying \mathcal{F} , one may find the “simplest” braid representative of the

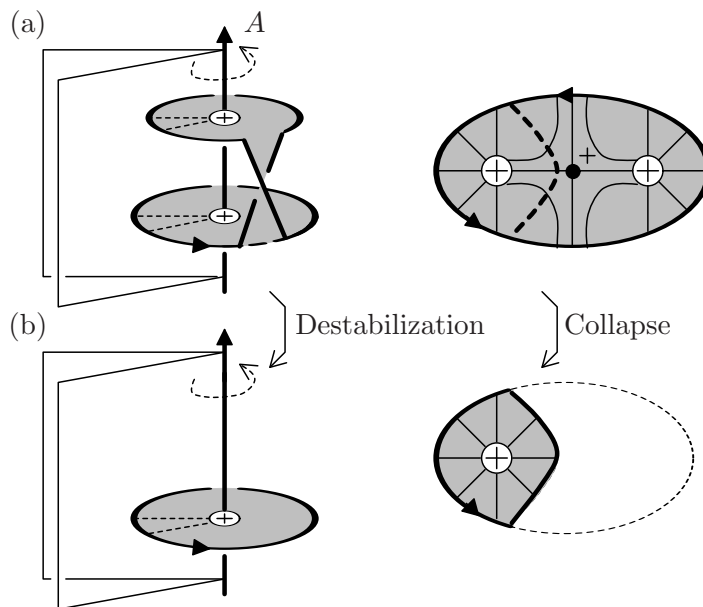


FIGURE 1. Example: Braid foliation and destabilization.

topological type of L and a sequence of operations on closed braids that simplifies L as well.

The braid foliation has vast applications to the study of knots and links in S^3 [4, 5, 6, 7, 8, 9, 10, 11]. Moreover, via the correspondence between the transverse links in the standard contact S^3 and the closed braids, the braid foliation is useful to solve problems in contact geometry, in particular, transversely (non-)simple knots [12, 13, 14].

Among them, here we highlight two results. In [10] Birman-Menasco classify the 3-braids: Two closed 3-braids representing topologically the same link are related to each other by one (either positive or negative) *flype* move. This is the key to their construction of transversely non-simple 3-braid links in [12]. In [7] they proved that every closed braid representative of the unknot can be deformed into the one-stranded braid by a sequence of exchange moves and destabilizations. Based on this, Birman-Wrinkle [14] gave an alternative topological proof of the theorem first proven by Eliashberg-Fraser [15]: The unknot in (S^3, ξ_{std}) is transversely simple.

It should be pointed out that the braid foliation is not so difficult to see or draw once embedding of the surface is understood. This contrasts strikingly with the flexibility of the characteristic foliations, which we describe next.

2.2. Characteristic foliation. Let (M, ξ) be a closed contact 3-manifold. Let $F \subset M$ be an oriented embedded surface, either closed or typically with Legendrian boundary for technical reasons. (A convex surface with transverse boundary is established by Etnyre and Van Horn-Morris [21, Section 2].) Integrating the vector field $\xi \cap TF$ on F , we get

a singular foliation $\mathcal{F}_\xi(F)$ on F , called the *characteristic foliation*. The characteristic foliation encodes information of the contact structure near F : If two contact structures induce the same characteristic foliation on F , then they are isotopic near F .

A surface F is called *convex* if there exists a vector field v whose flow preserves ξ and is transverse to F . For a convex surface F , the *dividing set* is a multi-curve on F defined by $\{p \in F \mid v_p \in \xi_p\}$. As the name suggests, it “divides” the characteristic foliation $\mathcal{F}_\xi(F)$. By Giroux’s flexibility theorem [25], it is the dividing set (not the whole characteristic foliation) that encodes essence of the contact structure near F : If two contact structures induce the same dividing set on F , then they are isotopic near F .

In [29], Honda invented *bypass attachment*, which allows us to modify dividing sets in controlled manner. With careful examination of the dividing sets (and the characteristic foliations) one can glue and/or cut contact 3-manifolds along convex surfaces, hence usual topological techniques apply. It lead to various results in contact topology including the non-existence of tight contact structures on a Poincaré homology sphere [18], and classification of the Legendrian knots of the $(2, 3)$ -cable of the $(2, 3)$ -torus knot and its transversely non-simplicity [20], both by Etnyre and Honda.

In practice, except for certain simple cases, it is not easy to grasp the entire picture of the characteristic foliations or the dividing sets, and not very clear how they change under isotopies of the surfaces. One of the merits of the open book foliation is to visualize the characteristic foliation in manageable way.

3. FOUNDATION OF OPEN BOOK FOLIATION

In this section we develop basic machineries of the open book foliation. As we indicate carefully, many of the ideas have sources in Birman-Menasco’s braid foliation theory. Often time the braid foliation provides motivating and the simplest examples for basic notions of the open book foliation. More advanced methods, such as generalization of the so-called exchange move in braid foliation theory, will be discussed in our subsequent paper [33].

3.1. Definition of open book foliation.

An *open book* (S, ϕ) is a pair of compact surface S with non-empty boundary ∂S and a mapping class $\phi \in \text{MCG}(S, \partial S)$. For an open book (S, ϕ) we define a closed oriented 3-manifold $M_{(S, \phi)}$ by

$$M_{(S, \phi)} = M_\phi \bigcup \left(\coprod_{|\partial S|} D^2 \times S^1 \right)$$

where M_ϕ denotes the mapping torus $S \times [0, 1]/(x, 1) \sim (\phi(x), 0)$, and the solid tori are attached so that for each point $p \in \partial S$ the circle $\{p\} \times S^1 \subset \partial M_\phi$ bounds a meridian disc for $D^2 \times S^1$. We say that (S, ϕ) is an *open book decomposition* of the 3-manifold $M = M_{(S, \phi)}$. A very simple example is $(S, \phi) = (D^2, \text{id})$ where $M_{(D^2, \text{id})} = S^3$. We view the union of the cores for the attached solid tori as an oriented fibered link B in M and call it the *binding* of the open book. Let $\pi : M \setminus B \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$ denote the fibration. The fibers $\pi^{-1}(t) = S_t$ where $t \in [0, 1)$ are called the *pages* of the open book.

An oriented link L in M is in *braid position* with respect to the open book (S, ϕ) if L is disjoint from the bindings and positively and transversely intersects each page S_t . This generalizes the familiar concept of braid position for a link in $M_{(D^2, \text{id})} = S^3$.

Let $\xi = \ker \alpha$ be a contact structure on M . We choose an open book (S, ϕ) that is compatible with (M, ξ) under the above-mentioned Giroux correspondence. Thus, $\alpha > 0$ on the binding B and $d\alpha$ is a positive area form on each page S_t .

Let F be an oriented connected compact surface smoothly embedded in M whose non-empty ¹ boundary ∂F is a transverse link in (M, ξ) . Thanks to Bennequin [1] and Pavelescu [43], any transverse link in (M, ξ) can be transversely isotoped to a closed braid in (S, ϕ) . Hence from now on, we may assume that ∂F is in braid position relative to (S, ϕ) .

Consider the singular foliation $\mathcal{F} = \mathcal{F}(F)$ on F induced by the fibers $\{S_t \mid t \in S^1\}$. That is, \mathcal{F} is a foliation obtained by integrating the singular line field $TS_t \cap TF$. We call each connected component of the intersection $F \cap S_t$ a *leaf*. By general position argument, the surface F can be perturbed, with the transverse link type of ∂F fixed, so that F satisfies the following four conditions, cf. [3, p.271].

- (OF i): The binding B pierces the surface F transversely in finitely many points. Moreover, each point $p \in B \cap F$ is an *elliptic* singularity in the foliation \mathcal{F} . See [24, p.166] for the definition of elliptic points. Geometrically, there exists a disc neighborhood $N_p \subset F$ of p on which the foliation $\mathcal{F}(N_p)$ is radial with the node p . The converse also holds: any elliptic singularity of \mathcal{F} is a transverse intersection point of B and $\text{Int}(F)$.
- (OF ii): There exists a tubular neighborhood $N \subset M$ of ∂F such that each leaf of the foliation $\mathcal{F}(F \cap N)$ transversely intersects ∂F .
- (OF iii): All the fibers S_t but finitely many intersect F transversely. Each exceptional fiber is tangent to $\text{Int}(F)$ at a single point. In particular, \mathcal{F} has no saddle-saddle connections.
- (OF iv'): The type of a tangency in (OF iii) is either saddle, local maximum, or local minimum.

Definition 3.1. [11, p.439] We say that a page S_t is *regular* if S_t intersects F transversely and is *singular* otherwise. Similarly, a leaf l of \mathcal{F} is called *regular* if l does not contain a tangency point and it is called *regular* otherwise.

The arguments in [3, p.272-273] imply the following:

Proposition 3.2. *Since ∂F is in braid position, no regular leaf of $\mathcal{F}(F)$ has both of its endpoints on ∂F . Hence, the regular leaves of \mathcal{F} are classified into the following three types:*

- a-arc* : An arc where one of its endpoints lies on B and the other lies on ∂F .
- b-arc* : An arc whose endpoints both lie on B .
- c-circle* : A simple closed curve.

¹Most of the arguments below hold even when F is closed.

Definition 3.3. We say that the singular foliation $\mathcal{F}(F)$ is an *open book foliation* if the above conditions **(OF i)**, **(OF ii)** and **(OF iii)** and the following condition **(OF iv)**, which is evidently stronger than **(OF iv')**, are satisfied and denote it by $\mathcal{F}_{ob}(F)$.

(OF iv): All the tangencies of F with fibers are saddles, and each corresponds to a hyperbolic point of $\mathcal{F}(F)$. See [24, p.166] for the definition of hyperbolic points.

Remark 3.4. The definitions of the braid foliation and the open book foliation are similar. However there are clear differences: (i) The ambient manifold for the braid foliation is S^3 , whereas the open book foliation is defined for any closed 3-manifolds. (ii) In braid foliation theory it requires that each regular leaf $l \subset S_t$ is an *essential* arc or loop in $S_t \setminus (S_t \cap L)$, as stated in [3, Theorem 1.1]. But this is not the case for the open book foliation, and it even allows F to be compressible.

The open book foliation is intrinsic in the following sense:

Theorem 3.5. *By isotopy that fixes the transverse link type of the boundary ∂F , every surface F admits an open book foliation $\mathcal{F}_{ob}(F)$. Moreover, we can make $\mathcal{F}_{ob}(F)$ with no c -circles.*

In order to prove Theorem 3.5, which will be done in Section 3.1.2, we first establish the sign of a singularity, by extending [3, p.280].

3.1.1. Sign of singularity. The open book foliation \mathcal{F}_{ob} has two kinds of singularities: An elliptic point which is a point of $B \cap F$, and a hyperbolic point where F is tangent to a fiber S_t . We say that an elliptic singularity p is *positive* (resp. *negative*) if the binding B is positively (resp. negatively) transverse to F at p . The sign of the hyperbolic singular point p is *positive* (resp. *negative*) if the positive normal direction of F at p agrees (resp. disagrees) with the direction of t . See Figure 2, where we describe an elliptic point by a hollowed circle with its sign inside, a hyperbolic point by a black dot with the sign indicated nearby, and positive normals to F , \vec{n}_F , by dashed arrows.

With this definition, we observe that:

Claim 3.6. *The elliptic point at the end of any a -arc is positive, and the endpoints of every b -arc have opposite signs.*

We see a hyperbolic singularity as a process of switching the configuration of leaves. As t increases, (possibly the same) leaves l_1 and l_2 approach along an arc γ , dashed in Figure 3, connecting l_1 and l_2 . At a critical moment, these two leaves form a hyperbolic singularity, then the configuration of the leaves is altered. See the passage in Figure 3. Thus, up to isotopy, a hyperbolic singularity is determined by γ . We call γ a *description arc* of a hyperbolic singularity.

Definition 3.7. We denote the number of positive (resp. negative) elliptic points of $\mathcal{F}_{ob}(F)$ by $e_+ = e_+(\mathcal{F}_{ob}(F))$ (resp. $e_- = e_-(\mathcal{F}_{ob}(F))$). Similarly, the number of positive (resp. negative) hyperbolic points is denoted by $h_+ = h_+(\mathcal{F}_{ob}(F))$ (resp. $h_- = h_-(\mathcal{F}_{ob}(F))$).

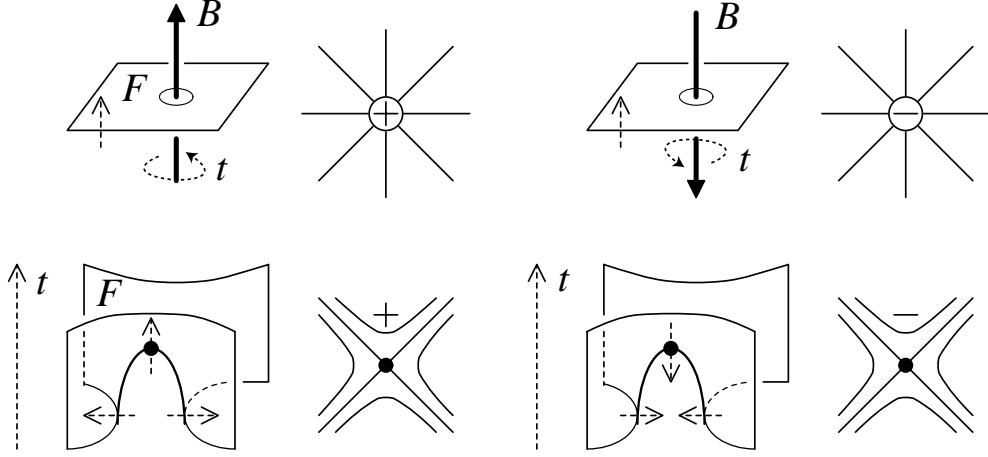
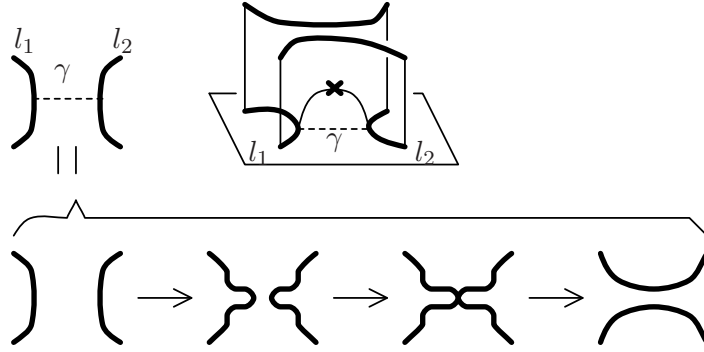
FIGURE 2. Signs of the singularities in \mathcal{F}_{ob} and normal vectors \vec{n}_F .

FIGURE 3. A description arc (dashed) for a hyperbolic singularity.

Proposition 3.8. *The Euler characteristic of the surface F has*

$$\chi(F) = (e_+ + e_-) - (h_+ + h_-).$$

To prove Proposition 3.8, we introduce *orientations* for leaves here: Both the surface F and the ambient manifold M are oriented so that the positive normal \vec{n}_F of F is canonically defined (cf. the dashed arrows in Figure 2). Then \vec{n}_F induces a transverse orientation of $\mathcal{F}_{ob}(F)$. The transverse orientation naturally introduces an orientation of leaves of $\mathcal{F}_{ob}(F)$ as follows. We orient each leaf of $\mathcal{F}_{ob}(F)$, for both regular and singular, so that if we stand up on the positive side of F and would walk along a leaf, the positive side of the corresponding page S_t of the open book would be on our left. In other words, at a non-singular point p on a leaf $l \subset (S_t \cap F)$ let \vec{v} be a positive tangent vector to l , then $\vec{n}_F \times \vec{v}$ is

a positive normal vector to the page S_t . As a consequence, positive/negative elliptic points become sources/sinks of the vector field \vec{v} .

Proof of Proposition 3.8. The orientation of the leaves induces a vector field on F . Since, by definition (OF iv), any singularity of $\mathcal{F}_{ob}(F)$ is either elliptic or hyperbolic, the equality follows from Poincaré-Hopf theorem. \square

3.1.2. *Proof of Theorem 3.5.* Since we do not assume the incompressibility for the surface F , we cannot directly apply Roussarie-Thurston's general position theorem [45, Theorem 4] or the proof of the corresponding result in braid foliation [3, Theorem 1.1]. Instead, we use the so-called finger move.

Proof. Let p be a local extremal tangency in the page S_t . Take an arc γ in S_t which connects p and $B = \partial S_t$. If γ intersects some leaves of $\mathcal{F}_{ob}(F)$, by small perturbation we make the intersections transverse. Now we push the part(s) of F in a neighborhood of γ by the *finger move* to remove the local extremal tangency. See Figure 4.

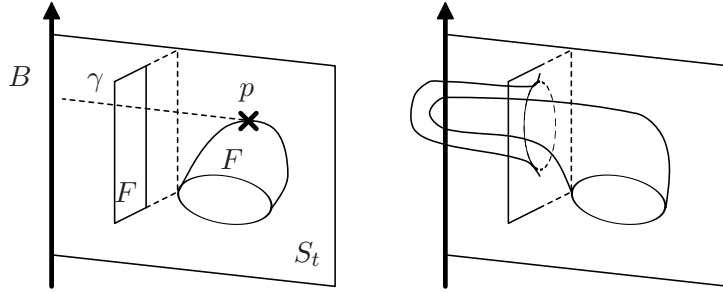


FIGURE 4. Finger move.

This operation introduces new elliptic and hyperbolic singularities. If the positive normal vector of S_t agrees (resp. disagrees) with the positive normal vector for F at p , then the finger move introduces one negative (resp. positive) hyperbolic point and a pair of \pm elliptic points. See the top sketch in Figure 5. For other part of F that is involved in the finger move, a pair of \pm elliptic points and a pair of \pm hyperbolic points are introduced. See the bottom row of Figure 5.

To confirm the second assertion, we apply a similar technique. By the above argument, we may assume that $\mathcal{F}_{ob}(F)$ has no local minimum or maximum. Suppose that S_{t_0}, S_{t_1} ($t_0 < t_1$) are singular pages such that for any $t \in (t_0, t_1)$, S_t contains a c -circle c_t and the set $A = \{c_t \mid t_0 < t < t_1\}$ forms a smooth annulus. Each of the limit circles $c_{t_i} \subset S_{t_i}$ ($i = 0, 1$) has one or two hyperbolic point(s). (In the latter case the two points are identified due to (OF iii).) Pick a properly embedded arc $\alpha \subset A$ that joins a non-singular point on c_{t_0} and a non-singular point on c_{t_1} and intersects all c_t ($t_0 < t < t_1$) transversely. Take an open rectangular neighborhood $R \subset F$ of α such that $\mathcal{F}_{ob}(R)$ has no singularities and is foliated by parallel arcs oriented in the same direction. See Sketch (1) of Figure 6.

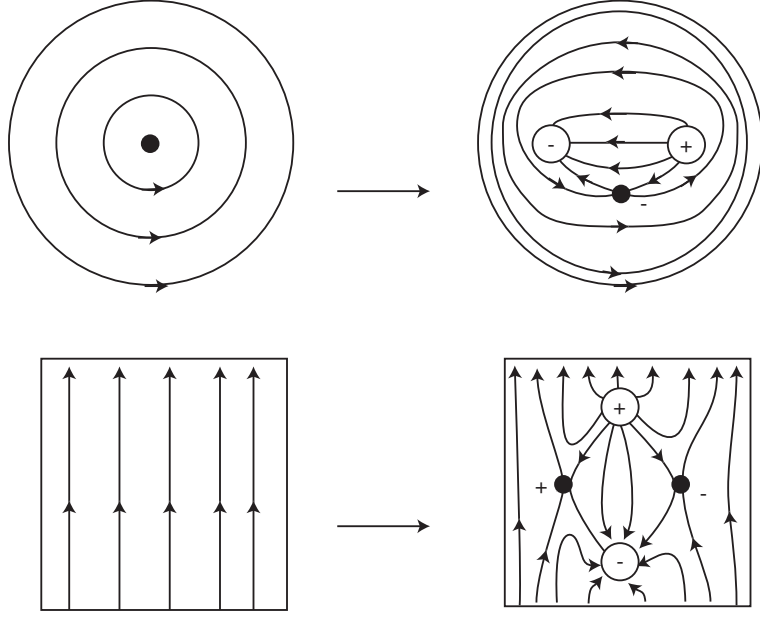


FIGURE 5. Foliation change before and after a finger move near a local maximum (top), and non-singular surface involved in the move (bottom).

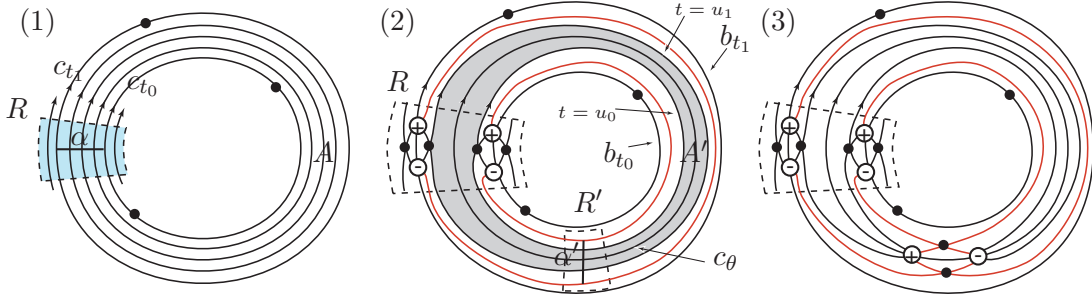


FIGURE 6. Removal of c -circles.

We apply the above finger move (the bottom sketch of Figure 5) twice in the interior of R so that for each $i = 0, 1$, the limit circle c_{t_i} is replaced by a b -arc, which we call $b_{t_i} \subset S_{t_i}$. Hence two pairs of \pm elliptic points and two pairs of \pm hyperbolic points are added to the region R . See Sketch (2) of Figure 6.

Now a new sub-annulus $A' = \{c_t \mid u_0 < t < u_1\} \subset A$, for some $t_0 < u_0 < u_1 < t_1$, is created so that each component of $\partial A'$ contains one of the new hyperbolic points in R and is part of a singular leaf in S_{u_i} that ends at the \pm elliptic points in R . Hence the boundary $\partial A'$ is better controlled than ∂A . We notice that $\text{Int}(A \setminus A')$ consists of b -arcs, i.e., no c -circles.

Choose a neighborhood $N_{A'} \subset A$ of A' (the shared region between the two red b -arcs in Sketch (2)) so that $N_{A'} \setminus A$ consists of b -arcs between the new elliptic points in R . Take a properly embedded arc α' in $N_{A'}$ that joins the boundaries of $N_{A'}$, intersects $\mathcal{F}_{ob}(N_{A'})$ transversely and does not contain any singularity of $\mathcal{F}_{ob}(N_{A'})$. Let $R' \subset A$ be a rectangular neighborhood of α' that is foliated by parallel arcs oriented in the same direction. Apply the finger mover in the interior of R' so that one c -circle, $c_\theta \subset A'$ where $u_0 < \theta < u_1$, is replaced by a pair of \pm elliptic points and two b -arcs, and a pair of \pm hyperbolic points is introduced at the ends of α' . See Sketch (3) of Figure 6. After the finger move, there is no c -circles in A' . \square

3.2. Region decomposition and negativity graph.

3.2.1. Region decomposition. Recall the three types of regular leaves: Type a, b and c (Proposition 3.2). The hyperbolic singularities in \mathcal{F}_{ob} are classified into six types, according to types of the regular leaves near the separatrices: Type aa, ab, bb, ac, bc , and cc as depicted in Figure 7. The corresponding notions in the braid foliation can be found in [3, p.279], where the ac -annulus is excluded.

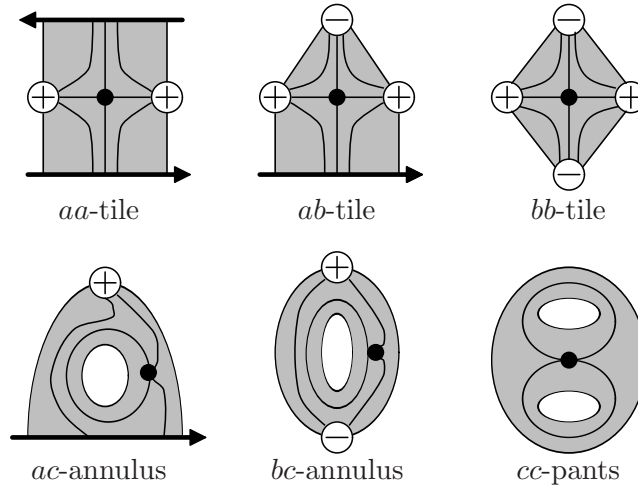


FIGURE 7. Six types of the region neighborhoods for hyperbolic singularities.

We call such a model neighborhood a *region neighborhood* or simply *region*. For each type, the sign of the hyperbolic point can be either \pm , although the sign assignments to the elliptic points are canonically determined due to Claim 3.6. For type ac - and bc -regions, the hyperbolic point can be on the left part of the annulus. Each region R is a closed set and $\text{Int}(R)$ is embedded in F . The topological type of $\text{Int}(R)$ is either disc, annulus, or pants. the elliptic points in an aa -, ab - or bb -tile are required to be distinct, since the separatrices of the tile sit on a single page of the open book. In general, R might be *degenerated*, that is, some parts of ∂R are identified in F , see Figure 8 for examples.

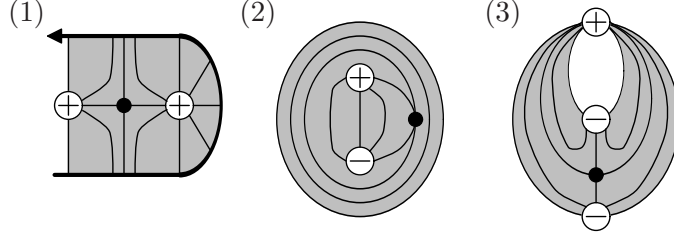


FIGURE 8. (1, 2) Degenerate regions. (3) An impossible degenerate region.

The next proposition, which generalizes [3, Theorem 1.2], demonstrates one of the useful features of the open book foliations.

Proposition 3.9 (Region decomposition). *If $\mathcal{F}_{ob}(F)$ contains a hyperbolic point, the surface F is decomposed into a union of regions whose interiors are disjoint.*

The decomposition is called a *region decomposition* of F . It provides a combinatorial description how F is embedded in $M_{(S,\phi)}$. We remark that if only aa -, ab -, and bb -tiles exist in the region decomposition, which is realizable by Proposition 3.5, then the region decomposition is a cellular decomposition of F : the elliptic points are regarded as 0-cells, and the regions are regarded as 2-cells.

Proof. Take region neighborhoods for all the hyperbolic points of \mathcal{F}_{ob} so that the interiors are disjoint. Let R denote the union of the region neighborhoods. Suppose that $F \setminus R$ is non-empty. Let D be a connected component of $F \setminus R$. By Proposition 3.2, $\mathcal{F}_{ob}(D)$ consists of only one type of regular leaves, type a , b , or c , as depicted in Figure 9. Since

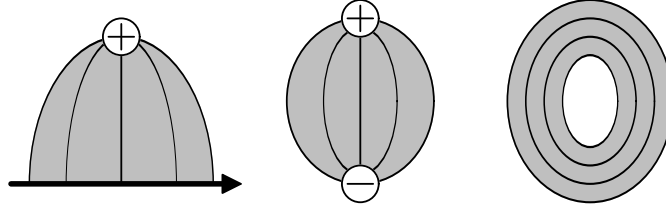


FIGURE 9. Regions foliated by regular leaves.

D is maximal, it is adjacent to some region $D_0 \subset R$ of a hyperbolic point p . The union $D \cup D_0$ is also a neighborhood region of p , so we extend R to $R \cup D$. Since F is compact, the number of the connected components of $F \setminus R$ is finite. Repeating the above procedure finitely many times, we obtain that $R = F$, i.e., F has a region decomposition. \square

3.2.2. *Negativity graph.* Finally, we introduce a key notion, the *negativity graph*.

The two flow lines, induced by the orientation vector field for \mathcal{F}_{ob} (Section 3.1.1), approaching to (resp. departing from) the hyperbolic point in an aa -, ab -, or bb -tile is called *stable* (resp. *unstable*) *separatrices*.

Definition 3.10. The *negativity graph* $G_{-,-}$ is a graph properly embedded in F . The edges of $G_{-,-}$ are the unstable separatrices for the negative hyperbolic points in $\mathcal{F}_{ob}(F)$. We view these negative hyperbolic points as part of the edges. The vertices of $G_{-,-}$ consists of the negative elliptic points in $\mathcal{F}_{ob}(F)$ and the end points of the edges of $G_{-,-}$ that sit on ∂F , called the *fake vertices*. See Figure 10.

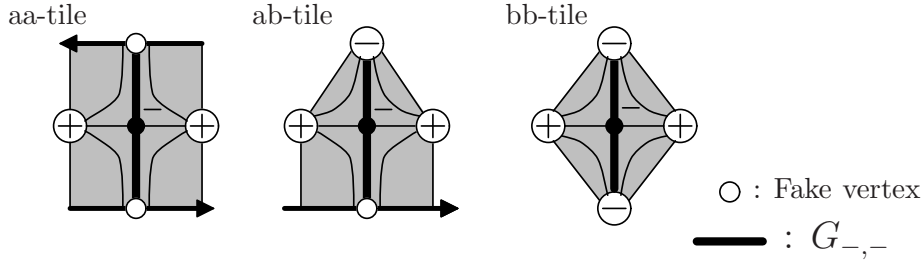


FIGURE 10. The negativity graph $G_{-,-}$.

Remark 3.11. In braid foliation theory, graphs $G_{\pm,\pm}, G_{\pm,\mp}$ [3, p.314], in which our negativity graph obviously has its origin, play important roles. In convex surface theory, $G_{-,-}$ can be viewed as a sub-graph of Giroux's graph [26, p.646].

3.3. Examples of the open book foliation.

Example 3.12. First we consider the simplest open book (D^2, id) which supports the standard tight contact structure on S^3 . This is the case that Birman and Menasco studied in their theory of braid foliations. Consider a 2-sphere F embedded as shown in the left sketch of Figure 11. Since F intersects the binding in four points, the open book foliation $\mathcal{F}_{ob}(F)$ has four elliptic points, two positive and two negative. It also has two hyperbolic points of opposite sign where F is tangent to pages of the open book. The right sketch of Figure 11 depicts the whole picture of $\mathcal{F}_{ob}(F)$. Note that $\mathcal{F}_{ob}(F)$ is not realizable as the braid foliation because it contains inessential b -arcs.

Example 3.13. Next we study a more complicated but informative example. Let us consider the open book (A, T_A^{-1}) where A denotes the annulus and T_A is the right-handed Dehn twist along the core of A . The ambient manifold is again S^3 with the negative Hopf link binding. However in this case (A, T_A^{-1}) supports an overtwisted contact structure.

In general, including this case, it is not easy to visualize the open book foliation immediately. To overcome this difficulty, we cut the complement of the binding $M_{(S,\phi)} \setminus B$ along the fiber S_0 . The resulting manifold is $S \times [0, 1]$ and each page S_t is naturally identified

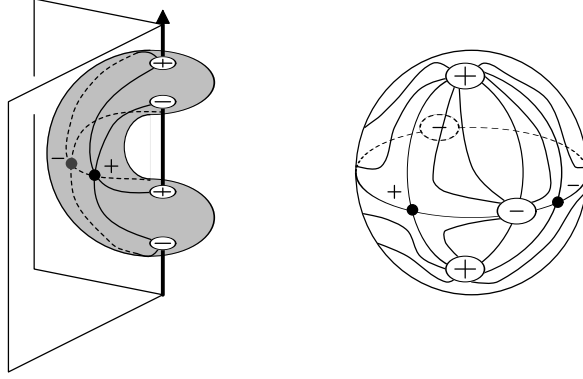
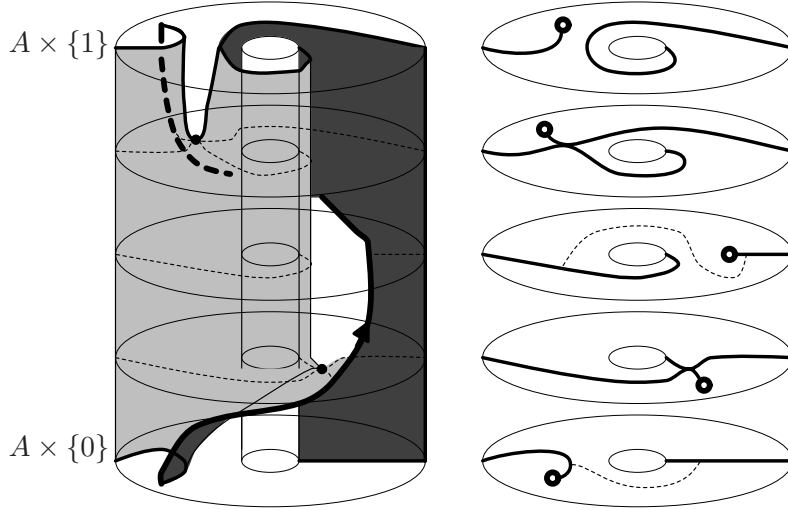


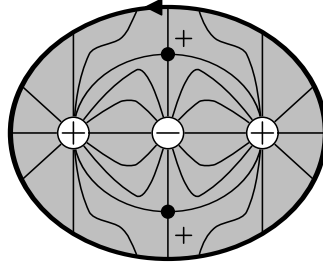
FIGURE 11. Example 3.12.

with $S \times \{t\}$. A surface $F \subset M_{(S,\phi)}$ is also cut out along $F \cap S_0$ and becomes a properly embedded surface, F' , in $S \times [0, 1]$ such that $F' \cap (S \times \{0\}) = \phi(F' \cap (S \times \{1\}))$ and $F' \cap (\partial S \times [0, 1]) = (F \cap \partial S_0) \times [0, 1]$.

In this manner, now we visualize an overtwisted disc D embedded in (A, T_A^{-1}) as in the left sketch of Figure 12. The right sketch depicts the sections of D by five fibers. We see that the multi-curve $D \cap (A \times \{1\})$ in the top annulus is identified with $D \cap (A \times \{0\})$ in the bottom under the monodromy T_A^{-1} . The right sketch also shows that the open book

FIGURE 12. Example 3.13. An overtwisted disc in an annulus open book (A, T_A^{-1}) .

foliation $\mathcal{F}_{ob}(D)$ contains two positive hyperbolic points, one positive elliptic point and one negative elliptic point. See Figure 13 for the entire picture of $\mathcal{F}_{ob}(D)$.

FIGURE 13. Example 3.13. The open book foliation $\mathcal{F}_{ob}(D)$.

4. OPEN BOOK FOLIATION VS. CHARACTERISTIC FOLIATION

Let $\mathcal{F}_\xi(\Sigma)$ denote the characteristic foliation of a surface Σ embedded in (M, ξ) . In this section, we compare the open book foliation and the characteristic foliation.

Theorem 4.1 (Structural stability theorem). *Assume that a surface Σ in (S, ϕ) admits the open book foliation $\mathcal{F}_{ob}(\Sigma)$. Then there exists a surface Σ' a C^1 -small perturbation of Σ fixing the boundary $\partial\Sigma$ pointwise such that $e_\pm(\mathcal{F}_{ob}(\Sigma')) = e_\pm(\mathcal{F}_\xi(\Sigma'))$ and $h_\pm(\mathcal{F}_{ob}(\Sigma')) = h_\pm(\mathcal{F}_\xi(\Sigma'))$.*

Moreover, if $\mathcal{F}_{ob}(\Sigma)$ contains no c -circles, then we may choose Σ' so that $\mathcal{F}_{ob}(\Sigma')$ and $\mathcal{F}_\xi(\Sigma')$ are topologically conjugate, namely there exists a foliation preserving homeomorphism of Σ' , and that Σ' is a convex surface.

Proof. Following the Thurston-Winkelenkemper construction [44], [24, p.152] of a contact structure compatible with the open book (S, ϕ) , we may assume that, away from the bindings, our contact 1-form is written as $\alpha = \beta_t + Cdt$ where $C \gg 1$ is a sufficiently large constant number and $\{\beta_t\}$ is a smooth family of 1-forms on the pages S_t such that the differential $d\beta_t$ is an area form of S_t of total area 2π and $\beta_1 = \phi^*\beta_0$.

Let $p \in \text{Int}(\Sigma)$ be a non-singular point in $\mathcal{F}_{ob}(\Sigma)$. Take a small open 3-ball neighborhood $U \subset M$ of p so that $\mathcal{F}_{ob}(U \cap \Sigma)$ has no singularity. We choose a coordinates (x, y, t) for U such that t is the coordinate for the interval $[0, 1]$ of the open book, $p = (0, 0, 0)$, and $U \cap \Sigma = \{(x, y, t) | t = y\}$. Note that $\mathcal{F}_{ob}(U \cap \Sigma)$ is induced by the constant vector field ∂_x . Suppose that the contact plane at a point $p = (x, y, t) \in U$ is given by

$$(4.1) \quad \xi_q = \text{span}\langle \partial_x + f(q)\partial_t, \partial_y + g(q)\partial_t \rangle_{\mathbb{R}}.$$

Then we have $0 = \alpha_q(\partial_x + f(q)\partial_t) = \beta_q(\partial_x) + Cf(q)$ and $0 = \alpha_q(\partial_y + g(q)\partial_t) = \beta_q(\partial_y) + Cg(q)$, hence $f(q) = -\beta_q(\partial_x)/C$ and $g(q) = -\beta_q(\partial_y)/C$. That is,

$$(4.2) \quad \beta_q = -C(f(q)dx + g(q)dy).$$

The assumption $C \gg 1$ implies that $|g| \ll 1$. Hence the normals $\vec{n}_\xi = (1, 0, f) \times (0, 1, g) = (-f, -g, 1)$ for ξ_q and $\vec{n}_\Sigma = (0, -1, 1)$ for $T_q\Sigma$ are not parallel, i.e., the point q is not a singularity of $\mathcal{F}_\xi(\Sigma)$.

Let $p \in \mathcal{F}_{ob}(\Sigma)$ be a positive hyperbolic point. (For the negative case, a parallel argument holds.) Take a small open 3-ball neighborhood $U \subset M$ of p so that p is the only singularity in $\mathcal{F}_{ob}(U \cap \Sigma)$. We choose coordinates (x, y, t) for U where t comes from the open book and U is the unit ball with center $p = (0, 0, 0)$. By a C^1 -small perturbation of Σ that preserve the topological structure of $\mathcal{F}_{ob}(\Sigma)$, we may assume that $U \cap \Sigma$ satisfies the equation

$$F(x, y, t) := t - xy|y| = 0.$$

The leaves of $\mathcal{F}_{ob}(U \cap \Sigma)$ are the flow lines of the vector field

$$X_{ob}(x, y) = 2x\partial_x - y\partial_y.$$

Note that the divergence at the singularity is $\text{div}(X_{ob})(\mathbf{0}) = 2 - 1 = 1 > 0$.

For each point $q \in U \cap \Sigma$, we assume (4.1), hence (4.2), again. Moreover, we may choose the 1-form β_q so that

$$(4.3) \quad f(q) = \frac{y+1}{C^2}, \quad g(q) = \frac{-x+1}{C^2}.$$

In particular, β_q is nowhere vanishing on $U \cap \Sigma$ and $d\beta_q = \frac{2}{C}dxdy > 0$. Let $\vec{n}_\xi = (-f(q), -g(q), 1)$ a positive normal vector for ξ_q , and $\vec{n}_\Sigma = \nabla F = (-y|y|, -2x|y|, 1)$ a positive normal for $T_q\Sigma$. Since $C \gg 1$, there is a unique solution $q_0 = (x_0, y_0, t_0) \in U \cap \Sigma$ for the equation $\vec{n}_\xi = \vec{n}_\Sigma$, i.e.,

$$(f(q_0), g(q_0)) = (y_0|y_0|, 2x_0|y_0|).$$

This implies that $y_0 \neq 0$. For, if $y_0 = 0$ then $(f(q_0), g(q_0)) = (0, 0)$ and $\beta_{q_0} = 0$ by (4.2), which is a contradiction.

Next we find a vector field $X_\xi(x, y)$ whose flow lines yield the characteristic foliation $\mathcal{F}_\xi(U \cap \Sigma)$. The vector $\vec{n}_\xi \times \vec{n}_\Sigma = (-g + 2x|y|, f - y|y|, |y|(2xf - yg))$ is tangent to the flow lines. Let

$$X_\xi(x, y) = \left(-\frac{g}{|y|} + 2x\right)\partial_x + \left(\frac{f}{|y|} - y\right)\partial_y.$$

Since $y_0 \neq 0$, we can take a small open disk neighborhood $U' \subset \{t = 0\}$ of (x_0, y_0) that does not contain $(0, 0)$. Then $X_\xi(x, y)$ is defined on U' . Clearly, $X_\xi(q_0) = 0$. The Jacobian matrix of X_ξ is the following:

$$\begin{bmatrix} \frac{-1}{|y|} \frac{\partial g}{\partial x} + 2 & \frac{g}{y|y|} - \frac{1}{|y|} \frac{\partial g}{\partial y} \\ \frac{1}{|y|} \frac{\partial f}{\partial x} & -\frac{f}{y|y|} + \frac{1}{|y|} \frac{\partial f}{\partial y} - 1 \end{bmatrix} \stackrel{(4.3)}{=} \begin{bmatrix} \frac{-1}{C^2|y|} + 2 & \frac{-x+1}{C^2y|y|} \\ 0 & -\frac{y+1}{C^2y|y|} + \frac{1}{C^2|y|} - 1 \end{bmatrix}$$

Since $C \gg 1$, at (x_0, y_0) it is approximated to $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. This shows that q_0 is a hyperbolic point of $\mathcal{F}_\xi(\Sigma)$. Moreover we have

$$\text{div} X_\xi(q_0) \approx \text{div} X_{ob}(\mathbf{0}) = 1 > 0,$$

i.e., the sign of q_0 is positive.

Suppose $p \in \text{Int}(\Sigma)$ is an elliptic point of sign ϵ . This means that at p a binding component, say γ , transversely intersects Σ with sign ϵ . Take a disc neighborhood $D \subset \Sigma$ of p whose open book foliation $\mathcal{F}_{ob}(D)$ contains no other singularities. By the standard

neighborhood theorem of transverse knots (see [24, p.76] for example), γ has a tubular neighborhood with cylindrical coordinates (r, t, z) so that $\alpha = dz + r^2 dt$. We push down by a finger a very small neighborhood $D_0 \subset D$ of p along γ without touching other part of the surface Σ . See Figure 14. This can be done by a C^1 -small perturbation preserving

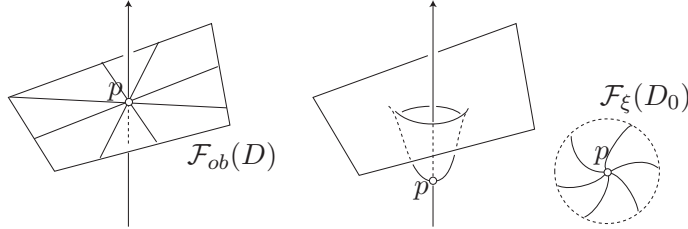


FIGURE 14.

$\mathcal{F}_{ob}(\Sigma)$. By the symmetry with respect to γ of the pushed D_0 and α , in the characteristic foliation $\mathcal{F}_\xi(D_0)$, the new p is also an elliptic point with the same sign ϵ .

The above argument concludes: $e_\pm(\mathcal{F}_{ob}(\Sigma)) = e_\pm(\mathcal{F}_\xi(\Sigma))$ and $h_\pm(\mathcal{F}_{ob}(\Sigma)) = h_\pm(\mathcal{F}_\xi(\Sigma))$.

For the second assertion, we observe that the open book foliation $\mathcal{F}_{ob}(\Sigma)$ is of Morse-Smale if $\mathcal{F}_{ob}(\Sigma)$ has no c -circles. Due to Peixoto, Morse-Smale vector fields are structurally stable [24, p.172]. Since the above argument shows that we may choose Σ' so that $\mathcal{F}_\xi(\Sigma')$ is sufficiently C^1 -close to $\mathcal{F}_{ob}(\Sigma)$, Peixoto's theorem implies that $\mathcal{F}_\xi(\Sigma')$ is topologically conjugate to $\mathcal{F}_{ob}(\Sigma)$. By [21, Lemma 2.1] Σ' is convex. \square

Remark 4.2. Proof of Theorem 4.1 shows that the open book foliation and the characteristic foliation may coincide, especially when there is no c -circles in \mathcal{F}_{ob} . It is interesting to compare the open book foliation and the characteristic foliations (on convex surfaces).

- For a given closed surface F , we can always find a convex surface F_{cv} that is isotopic and C^∞ -close to F . However, in general, there may not exist a surface that is C^1 -close to F and admitting an open book foliation.
- The *dividing set* Γ of a convex surface F encodes essential information of local contact structure near F . It defines a decomposition $F \setminus \Gamma = \Gamma_+ \sqcup \Gamma_-$ of F . If $\mathcal{F}_{ob}(F)$ has no c -circles then the region Γ_- is homotopy equivalent to our negativity graph $G_{-, -}$.
- In the characteristic foliation on a convex surface, any closed leaf is either repelling or attracting, and there is no type ac -, bc - and cc -hyperbolic singularities (cf. Figure 7) due to the Morse-Smale condition (cf. [24, p.171]). On the other hand, a neighborhood of a c -circle in an open book foliation is foliated by parallel c -circles.
- In the theory of convex surfaces, *Giroux elimination lemma* allows us to remove a pair of elliptic and hyperbolic singularities of the same sign by an arbitrary C^0 -small isotopy. Suppose that $\mathcal{F}_{ob}(\Sigma) = \mathcal{F}_\xi(\Sigma)$ (topologically conjugate) and that $\mathcal{F}_\xi(\Sigma)$ admits Giroux elimination to be deformed into \mathcal{F}_\star . However in general, \mathcal{F}_\star is not realizable as an open book foliation even with local perturbation of Σ . For

the open book foliations, removing singular points is much more subtle. We will discuss this issue in the subsequent paper [33].

5. THE SELF LINKING NUMBER

Using Pavelescu's theorem [43], we can identify a transverse link in (M, ξ) with a closed braid in a compatible open book (S, ϕ) . The goal of this section (Theorem 5.4) is to obtain a self-linking number formula for closed braids by constructing Seifert surfaces and counting the singularities in their open book foliations. Our formula contains a somewhat mysterious function $c : \text{MCG}(S, \partial S) \times H_1(S; \partial S) \rightarrow \mathbb{Z}$. After careful study of the function c we see in Theorem 5.17 that c is related to the first Johnson-Morita homomorphism, which is well-studied in mapping class group theory.

5.1. Self-linking number formula for braids.

By Theorem 4.1 along with the self-linking number formula in terms of the characteristic foliation, see [24, p.203] for example, we obtain the following formula in terms of our open book foliation:

Proposition 5.1. *Suppose that $F \subset M_{(S, \phi)}$ is a surface admitting an open book foliation \mathcal{F}_{ob} . In particular, ∂F is a transverse link in $(M_{(S, \phi)}, \xi_{(S, \phi)})$. Let e_{\pm}, h_{\pm} be integers as defined in Definition 3.7. Then*

$$sl(\partial F, [F]) = -\langle e(\xi), [F] \rangle = -(e_+ - e_-) + (h_+ - h_-).$$

To state Theorem 5.4, we fix notations and conventions: Let $S = S_{g,r}$ be an oriented genus g surface with r boundary components $\gamma_1, \dots, \gamma_r$. The orientation of γ_i is induced from that of S . Let b be an n -stranded braid in $S \times [0, 1]$ with $b \cap S_1 = b \cap S_0 = \{x_1, \dots, x_n\} \subset S$. By braid isotopy, we may assume that points x_1, \dots, x_n are lined up in this order on an arc parallel to and very close to γ_1 . The arc $\{x_i\} \times [0, 1]$ is called the i -th braid strand in $S \times [0, 1]$. We define oriented loops, $\rho_i \subset S$ ($i = 1, \dots, 2g + r - 1$) with the base point x_n as in Figure 15 and view them as generators of braids in $S \times [0, 1]$. Let σ_i denote the

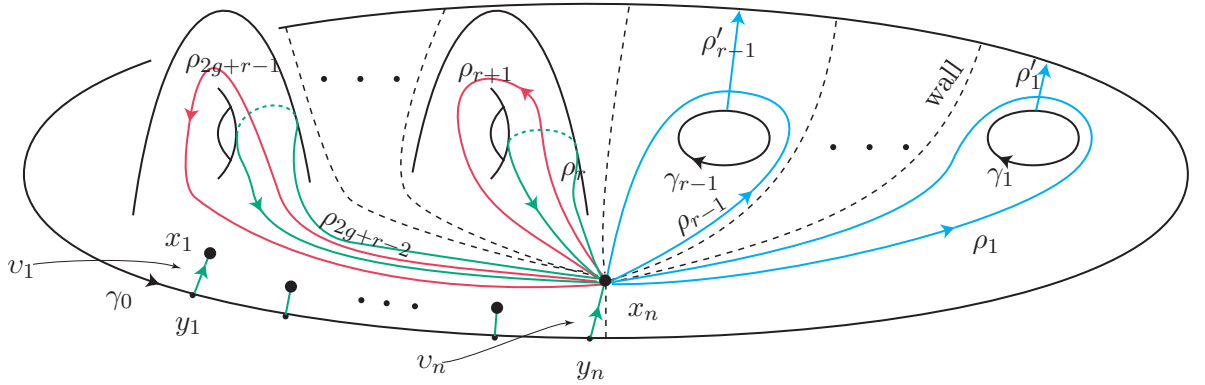


FIGURE 15. Surface S .

positive half twist of the i -th and the $(i+1)$ -th braid strands.

As a consequence of Birman exact sequence [2], the braid b is represented by a braid word $b_1^{\varepsilon_1} b_2^{\varepsilon_2} \cdots b_l^{\varepsilon_l}$ (read from the left) where $b_i \in \{\rho_1, \dots, \rho_{2g+r-1}, \sigma_1, \dots, \sigma_{n-1}\}$ and $\varepsilon_i \in \mathbb{Z} \setminus \{0\}$. Fix a diffeomorphism $\phi \in \text{Aut}(S, \partial S)$. Since x_i is near $\gamma_1 \subset \partial S$, we have $\phi(x_i) = x_i$ and identify $\{x_i\} \times \{1\}$ and $\{x_i\} \times \{0\}$ under ϕ that yields a closed braid \hat{b} in $M_{(S,\phi)}$. We assume that \hat{b} is null-homologous in the rest of the section.

Claim 5.2. Put $[b] = \sum_{i=1}^l \varepsilon_i [b_i] \in H_1(S; \mathbb{Z})$, where we set $[\sigma_k] = 0$ for $k = 1, \dots, n-1$. Then there exists (not necessarily uniquely) a homology class $a \in H_1(S, \partial S; \mathbb{Z})$ such that $[b] = a - \phi_*(a)$ in $H_1(S; \mathbb{Z})$.

Proof. The homology group of the manifold $M_{(S,\phi)}$ has been computed by Etnyre-Ozbagci [23, p.3136]:

$$H_1(M_{(S,\phi)}; \mathbb{Z}) = \langle [\rho_1], \dots, [\rho_{2g+r-1}] \mid [\rho'_i] - \phi_*[\rho'_i] = 0, \quad i = 1, \dots, 2g+r-1 \rangle,$$

where

$$\rho'_i = \begin{cases} \text{a properly embedded arc from } \gamma_0 \text{ to } \gamma_i \text{ and dual to } \rho_i, & \text{for } i = 1, \dots, r-1, \\ \rho_i, & \text{for } i = r, \dots, 2g+r-1. \end{cases}$$

Though ρ'_i is an arc for $i = 1, \dots, r-1$, since $\phi = \text{id}$ on ∂S , we can view $\rho'_i \cup \phi(-\rho'_i)$ as a (possibly non-simple) oriented loop in $\text{Int}(S)$. Then we consider $[\rho'_i] - \phi_*[\rho'_i]$ as an element of $H_1(S; \mathbb{Z})$ representing the loop $\rho'_i \cup \phi(-\rho'_i)$.

Since $[\hat{b}] = 0$ in $H_1(M; \mathbb{Z})$, there exist $s_i \in \mathbb{Z}$ for $i = 1, \dots, 2g+r-1$, such that

$$[b] = \sum_{i=1}^{2g+r-1} s_i ([\rho'_i] - \phi_*[\rho'_i]) \quad \text{in } H_1(S; \mathbb{Z}).$$

Hence if we put $a = \sum_{i=1}^{2g+r-1} s_i [\rho'_i]$, under the identification $[\rho'_i] - \phi_*[\rho'_i] = [\rho'_i \cup \phi(-\rho'_i)]$, we have $[b] = a - \phi_*(a)$. \square

Definition 5.3. For homology classes $[a_1] \in H_1(S, \partial S; \mathbb{Z})$ and $[a_2] \in H_1(S; \mathbb{Z})$ we denote the *algebraic intersection number* by $[a_1] \cdot [a_2] \in \mathbb{Z}$. It counts the transverse intersections of representatives a_1 and a_2 algebraically in the way described in Figure 16. For example, we have $[\rho'_1] \cdot [\rho_1] = 1$ and $[\rho_r] \cdot [\rho_{r+1}] = 1$.



FIGURE 16. Algebraic intersection number $[a_1] \cdot [a_2]$.

Theorem 5.4 (Self linking number formula). *Let $[b] \in H_1(S; \mathbb{Z})$ and \hat{b} be as above. Let $a \in H_1(S, \partial S; \mathbb{Z})$ be a homology class such that $[b] = a - \phi_*(a)$ in $H_1(S; \mathbb{Z})$. For a choice of a , there exists a Seifert surface Σ for \hat{b} such that the self-linking number is given by the formula:*

$$(5.1) \quad sl(\hat{b}, [\Sigma]) = -n + \widehat{\exp}(b) - \phi_*(a) \cdot [b] + c(\phi, a),$$

where

$$\widehat{\exp}(b) = \sum_{i=1}^l \varepsilon_i - \sum_{1 \leq j < i \leq l} \varepsilon_i \varepsilon_j [b_j] \cdot [b_i]$$

and

$$c : \text{MCG}(S, \partial S) \times H_1(S, \partial S; \mathbb{Z}) \rightarrow \mathbb{Z}$$

is a function.

Remark 5.5. An explicit description of the function c will be given in Theorem 5.17. The formula (5.1) is a generalization of Bennequin's self linking formula of braids in the open book (D^2, id) [1], and it also generalizes the works in [36] and [37].

Assume that $S = D^2$. Then $\widehat{\exp}$ is equal to the usual exponent sum, $\exp : B_n \rightarrow \mathbb{Z}$, for Artin braid group B_n , and $[b] \cdot \phi_*(a) = c(\phi, a) = 0$. Thus our formula (5.1) is just Bennequin's self linking formula

$$sl(\hat{b}) = -n + \exp(b).$$

With more elaborate investigation of the function c we will deduce the self-linking number formulae of [36], [37] in Section 5.3 below.

Proof. For each $i = 1, \dots, n$, take a point y_i on the binding γ_0 near x_i so that y_1, \dots, y_n lined up in this order with respect to the orientation of γ_0 , see Figure 15. Choose a properly embedded arc v_i from x_i to y_i that contained in a small collar neighborhood of γ_0 so that $\phi(v_i) = v_i$. We require that v_1, \dots, v_n are mutually disjoint.

To introduce 1-submanifolds $A_0, A_{1/2}, A_1 \subset S$, we divide the surface S by *walls* (dashed arcs in Figure 15) into $g + r - 1$ *chambers* so that g of which are once-punctured tori and $r - 1$ of which are annuli.

Let A_1 be the set of oriented curves

$$A_1 = v_1 \cup \dots \cup v_n \cup N(a)$$

such that

- (1) $[A_1] = [N(a)] = a$ in $H_1(S, \partial S; \mathbb{Z})$.
- (2) A_1 is properly embedded in S .
- (3) $N(a)$ does not intersect the walls.
- (4) Any subset of $N(a)$ has non-trivial homology in $H_1(S, \partial S; \mathbb{Z})$. In other words, the component of A_1 in a torus (resp. an annulus) chamber is a torus knot or link (resp. a set of parallel arcs joining γ_1 and γ_i) and oriented in the same direction.

For example, if $a = \sum_{i=1}^{2g+r-1} s_i [\rho'_i]$ as in the proof of Claim 5.2, then we can put $N(a) = \bigcup_{i=1}^{2g+r-1} s_i \rho'_i$ where $s_i \rho'_i$ means $|s_i|$ parallel copies of the curve ρ'_i (resp. $-\rho'_i$) and the intersections are smoothed.

Similarly, let $A_{1/2}$ be the set of oriented curves

$$A_{1/2} = v_1 \cup \cdots \cup v_n \cup N(\phi_* a)$$

such that $[A_{1/2}] = [N(\phi_* a)] = \phi_* a$ in $H_1(S, \partial S; \mathbb{Z})$ and parallel conditions to the above (2)–(4) are satisfied.

Lastly, we define $A_0 = \phi(A_1)$. Unlike A_0 or $A_{1/2}$, this A_0 possibly intersects the walls. In short, we have in $H_1(S, \partial S; \mathbb{Z})$,

$$[A_1] = a, \quad \text{and} \quad [A_{1/2}] = [A_0] = \phi_* a.$$

Now we construct an oriented surface Σ_* embedded in $S \times [0, 1/2]$ with $\Sigma_* \cap S_{1/2} = A_{1/2}$ and $\Sigma_* \cap S_0 = -A_0$. Let w be one of the walls. Since $[A_{1/2}] = [A_0]$ and $A_{1/2}$ does not intersect w , the algebraic intersection number $[A_0] \cdot w = 0$. We take a collar neighborhood $\nu(w) \subset S$ of w so that each component of $\nu(w) \cap A_0$ has geometric intersection number 1 with w . Arcs in $\nu(w) \cap A_0$ may have inconsistent orientations. As $t \in [0, 1/2]$ increases, as depicted in Figure 17, we apply the configuration changes to consecutive pairs of arcs



FIGURE 17. Configuration change of $\nu(w) \cap A_0$.

in $\nu(w) \cap A_0$ with opposite orientations, until we remove all the arcs in $\nu(w) \cap A_0$. Each configuration change introduces a new hyperbolic singularity. We repeat the procedure for all the walls. The deformed A_0 , which we denote A'_0 , no longer intersects the walls.

As bi-products, null-homologous sets of c -circles may be created, which we remove by the following three steps.

(Step 1): If there exist c -circles bounding concentric discs in a chamber H of S and oriented in the same direction, then we remove them from the outermost one. We can find a description arc of a hyperbolic point (cf. Figure 3) that joins the outermost c -circle and some curve in $A'_0 \cap H$ and properly embedded in $H \setminus A'_0$. As shown in the top row of Figure 18, a hyperbolic singularity is introduced, then the c -circle disappears. The sign of the hyperbolic singularity is $+$ if and only if the c -circle is oriented clockwise.

(Step 2): If there is a pair of c -circles with opposite orientations that bounds an annulus in $S \setminus A'_0$, then remove the pair by introducing a hyperbolic singularity, of sign ε , between the two c -circles as in Figure 18. The resulting c -circle bounds a disc that can be removed by Step 1 with the expense of another hyperbolic singularity of sign $-\varepsilon$.

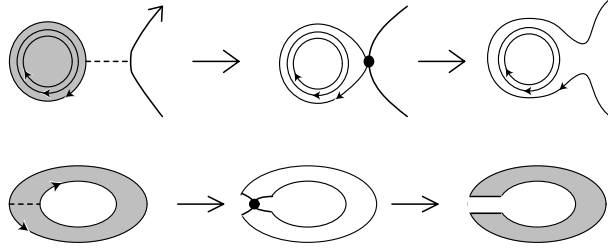
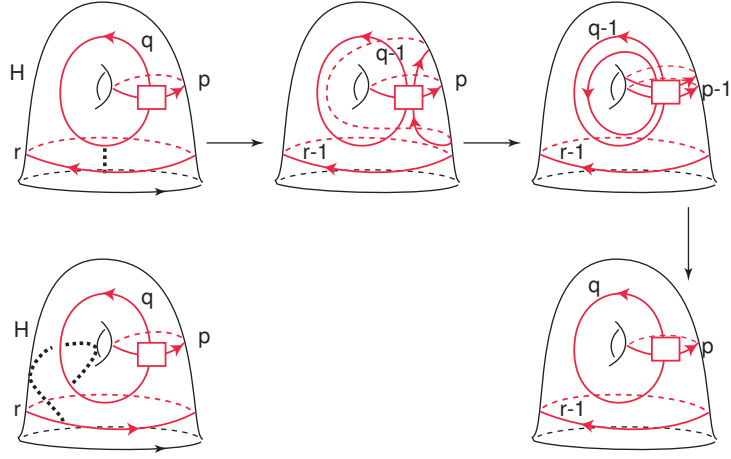


FIGURE 18. Step 1 (top) and Step 2 (bottom)

(Step 3): Let H be a once-punctured torus chamber of S separated by the walls. After Steps 1, 2, above, $A'_0 \cap H$ is a union of some (p, q) torus link and r many c -circles parallel to the boundary of H and oriented in the same direction. As in Figure 19, we remove the r many c -circles by introducing r many hyperbolic points of the same sign, which depends on the signs of p, q and the orientation of the boundary parallel c -circles.

FIGURE 19. Step 3. Remove boundary parallel null-homologous c -circles by configuration changes along the dashed arcs.

After applying Steps 1–3 to A'_0 , no subset of A'_0 is null-homologous, hence it has been deformed to $A_{1/2} \subset S_{1/2}$, up to isotopy. This yields the desired surface Σ_* . Let $c(\phi, a, \Sigma_*)$ be the value $h_+ - h_-$ for the surface Σ_* .

Claim 5.6. *The function $c(\phi, a, \Sigma_*)$ is indeed independent of the choice of configuration changes to $\nu(w) \cap A_0$ and how we apply Steps 1–3, that is, the construction of the surface Σ_* . Hence it justifies to rewrite it as $c(\phi, a)$.*

Proof. Let us take another sequence of configuration changes which determines another surface, say Σ'_* . We consider $-\Sigma'_*$ embedded in $S \times [1/2, 1]$. Then a \pm hyperbolic singular point in $\mathcal{F}_{ob}(\Sigma'_*)$ turns to a \mp hyperbolic point in $\mathcal{F}_{ob}(-\Sigma'_*)$.

We glue Σ_* and $-\Sigma'_*$ at the page $S_{1/2}$ and obtain a surface F' in $S \times [0, 1]$. Since $F' \cap S_1 = \phi(A_1) = -(F' \cap S_0)$, we can further identify $F' \cap S_1$ and $F' \cap S_0$ by the identity map that defines a surface F embedded in the open book (S, id) . Then F is a disjoint union of n discs pierced by the binding γ_1 and the rest of the components, which we denote F_* . In particular, all the hyperbolic singularities in $\mathcal{F}_{ob}(F)$ belong to F_* and

$$(5.2) \quad h_+(\mathcal{F}_{ob}(F_*)) - h_-(\mathcal{F}_{ob}(F_*)) = c(\phi, a, \Sigma_*) - c(\phi, a, \Sigma'_*).$$

In the course of constructing Σ_* above, no new elliptic points are introduced and all the elliptic points in $\mathcal{F}_{ob}(F_*)$ correspond to the endpoints of some ρ'_i ($i = 1, \dots, r-1$). Since each pair contributes elliptic points of opposite sign,

$$(5.3) \quad e_+(\mathcal{F}_{ob}(F_*)) = e_-(\mathcal{F}_{ob}(F_*)).$$

Let ξ_{id} be the contact structure supported by the open book (S, id) . Since the Euler class of ξ_{id} is equal to zero, by Proposition 5.1, (5.2) and (5.3),

$$0 = \langle e(\xi_{id}), [F_*] \rangle = -c(\phi, a, \Sigma_*) + c(\phi, a, \Sigma'_*).$$

This concludes the statement of the claim. \square

We continue proving Theorem 5.4.

The next goal is to construct an oriented surface Σ_{**} embedded in $S \times [1/2, 1]$ with $\Sigma_{**} \cap S_1 = A_1$ and $\Sigma_{**} \cap S_{1/2} = -A_{1/2}$. Recall that b is represented by the braid word $b_1^{\varepsilon_1} \dots b_l^{\varepsilon_l}$. Let $I_i = [\frac{l+i-1}{2l}, \frac{l+i}{2l}]$ then $[1/2, 1] = I_1 \cup \dots \cup I_l$. We will build an oriented surface Σ_i embedded in $S \times I_i$ inductively from $i = 1$ to l such that

- (1) $\Sigma_1 \cap S_{1/2} = -A_{1/2}$ and $\Sigma_l \cap S_1 = A_1$,
- (2) $\Sigma_i \cap S_{(l+i)/2l} = -(\Sigma_{i+1} \cap S_{(l+i)/2l}) =: A_{(l+i)/2l}$,
- (3) $A_{(l+i)/2l}$ is properly embedded in $S_{(l+i)/2l}$ and does not intersect the walls,
- (4) $A_{(l+i)/2l}$ contains $v_1 \cup \dots \cup v_n$ and any subset of $A_{(l+i)/2l} \setminus (v_1 \cup \dots \cup v_n)$ has non-trivial homology in $H_1(S, \partial S)$,
- (5) $\partial \Sigma_i \cap (S \times \text{Int}(I_i)) = b_i^{\varepsilon_i}$, so $[A_{(l+i)/2l}] = [A_{1/2}] + \varepsilon_1[b_1] + \dots + \varepsilon_i[b_i]$ in $H_1(S, \partial S)$.

Then eventually we will define $\Sigma_{**} = \Sigma_1 \cup \dots \cup \Sigma_l$.

Suppose that we have constructed $\Sigma_1, \dots, \Sigma_{i-1}$ satisfying the above conditions.

(Case 1) If the braid word $b_i = \sigma_k$, then as $t \in I_i$ increases, apply the deformation of the graph $A_{(l+i-1)/2l}$, as in the passage of Figure 20, for $|\varepsilon_i|$ times that takes place in a small neighborhood of v_k and v_{k+1} . We call the trace of the graph Σ_i , which satisfies the above conditions. It has $|\varepsilon_i|$ hyperbolic singularities of $\text{sgn}(\varepsilon_i)$.

(Case 2-1) Suppose that $b_i = \rho_k$ and $\varepsilon_i = 1$. Let H be the chamber that ρ_k belongs to.

Assume that $r \leq k \leq 2g + r - 1$ so that H is a torus with connected boundary. For simplicity, put $u = A_{(l+i-1)/2l}$. By conditions (3), (4) above, we may assume that $u \cap H$ is some (p, q) -torus link. As $t \in I_i$ increases, move the point x_n along ρ_k . See Figure 21. To come back to the original position, x_n has to traverse u , which yields $p = [u] \cdot [\rho_k]$

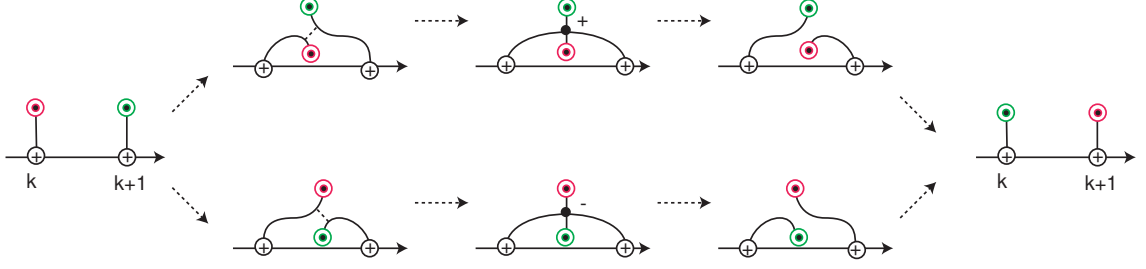


FIGURE 20. (Case 1) Deformation of graph $v_k \cup v_{k+1}$ corresponding to σ_k (top) and σ_k^{-1} (bottom).

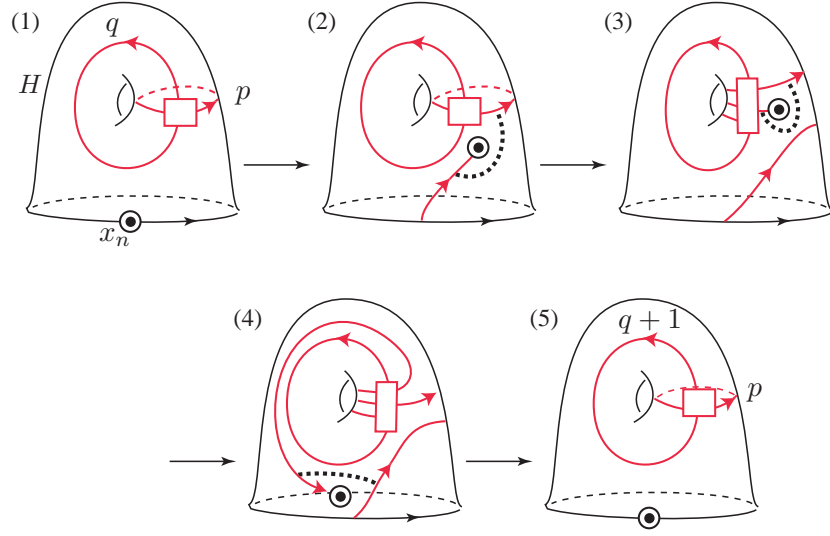


FIGURE 21. Construction of surface Σ_i when $b_i^{\varepsilon_i} = \rho_k$ (case 2-1).

many negative ($= -\text{sgn}(\varepsilon_i)$) hyperbolic points. Moreover, the last step (Sketch (4)) adds one more hyperbolic singularity of positive ($= \text{sgn}(\varepsilon_i)$) sign. This defines the surface Σ_i in $S \times I_i$. In summary, the value $h_+ - h_-$ increases by

$$\text{sgn}(\varepsilon_i) \cdot 1 - [u] \cdot [\rho_k]$$

and the class $[u] \in H_1(S, \partial S)$ is replaced by $[u] + [\rho_k]$ (compare Sketches (1) and (4)). No circle bounding a disc in S has been created.

When $k = 1, \dots, r-1$ (i.e., the chamber H is an annulus) a parallel argument holds and the value $h_+ - h_-$ increases by $\text{sgn}(\varepsilon_i) \cdot 1 - [u] \cdot [\rho_k]$. The only difference is that $[\rho_k] = 0$ in $H_1(S, \partial S)$, which causes no effect for the rest of the arguments.

(Case 2-2) If $b_i = \rho_k$ and $\varepsilon_i \neq 1, 0$, repeat the above construction $|\varepsilon_i|$ times. Since $([u] + [\rho_k]) \cdot [\rho_k] = [u] \cdot [\rho_k]$, the total change in $h_+ - h_-$ is

$$\begin{aligned}
 (5.4) \quad \varepsilon_i - \varepsilon_i[u] \cdot [\rho_k] &\stackrel{(5)}{=} \varepsilon_i - \varepsilon_i([A_{1/2}] + \varepsilon_1[b_1] + \cdots + \varepsilon_{i-1}[b_{i-1}]) \cdot [b_i] \\
 &= \varepsilon_i - \left(\sum_{j=1}^{i-1} \varepsilon_i \varepsilon_j [b_j] \cdot [b_i] \right) - \varepsilon_i [A_{1/2}] \cdot [b_i].
 \end{aligned}$$

After constructing $\Sigma_1, \dots, \Sigma_l$, we glue them and obtain the desired surface Σ_{**} , which contributes to the algebraic count of the hyperbolic singularities by

$$h_+ - h_- = \sum_{i=1}^l \varepsilon_i - \left(\sum_{i=1}^l \sum_{j=1}^{i-1} \varepsilon_i \varepsilon_j [b_j] \cdot [b_i] \right) - [A_{1/2}] \cdot [b].$$

Finally, we glue Σ_* and Σ_{**} by the identifications $\Sigma_* \cap S_{1/2} = -(\Sigma_{**} \cap S_{1/2})$ and $-(\Sigma_* \cap S_0) = \phi(\Sigma_{**} \cap S_1)$, which yields a Seifert surface Σ for \hat{b} . By the construction, it is clear that $y_1, \dots, y_n \in \gamma_0$ are positive elliptic points and the end points of arc ρ'_i are elliptic points with distinct signs. By Proposition 5.1, we obtain our self linking formula (5.1). \square

5.2. Properties of the function c .

In this section we study the mysterious function $c(\phi, a)$ where $\phi \in \text{MCG}(S, \partial S)$ and $a \in H_1(S, \partial S)$, that appears our self linking number formula (5.1).

We first introduce various notions. A homology class $a \in H_1(S, \partial S)$ is represented by a properly embedded oriented simple closed curves and arcs in S . Among such multi-curve representatives of a , we take a special one, A , which satisfies the following conditions:

- A does not intersect the walls.
- Any subset of A has non-trivial homology in $H(S, \partial S)$, i.e., the component of A in a torus (resp. an annulus) chamber is a torus knot or link (resp. a set of parallel arcs joining γ_1 and γ_i) and oriented in the same direction.

Clearly, such a representative A is uniquely determined up to isotopy, called the *normal form* of the homology class a , and is denoted by $N(a)$.

Definition 5.7. Let A and A' be multi-curve sets on S which represent the same homology class in $a \in H_1(S, \partial S)$. An *open book foliation cobordism* (OB cobordism, in short) between A and A' , denoted by $A \xrightarrow{\Sigma} A'$, is a properly embedded oriented compact surface Σ in $\text{Int}(S) \times [0, 1]$ such that:

- $\partial \Sigma = -A \cup A'$.
- $\Sigma \cap S_0 = -A$ and $\Sigma \cap S_1 = A'$.
- The fibration $\{S_t\}_{t \in [0, 1]}$ induces a foliation \mathcal{F}_Σ on Σ all of which singularities are of hyperbolic.

As we have seen, in the proof of Theorem 5.4, there is an OB cobordism $A \xrightarrow{\Sigma} N(a)$ for any multi-curve representative A of a . This implies that if two multi-curves A and A' represent the same homology class, then there exists an OB cobordism $A \xrightarrow{\Sigma} A'$.

For $A \xrightarrow{\Sigma} A'$, we define the integer $d(\Sigma)$ as the value $h_+(\mathcal{F}_\Sigma) - h_-(\mathcal{F}_\Sigma)$, where $h_+(\mathcal{F}_\Sigma)$ (resp. $h_-(\mathcal{F}_\Sigma)$) denotes the number of the positive (resp. negative) hyperbolic singular points of \mathcal{F}_Σ . As Claim 5.6 shows, $d(\Sigma)$ does not depend on the choice of Σ and but it only depends on the representatives A and A' . Thus, we may denote $d(A, A') = d(\Sigma)$.

With these notions, our function c in Theorem 5.4 is described as follows. Let $\phi \in \text{MCG}(S, \partial S)$ and $a \in H_1(S, \partial S)$, then

$$c(\phi, a) = d(\phi N(a), N(\phi_* a)).$$

Note, in general, the multi-curves $\phi(N(a))$ may not be isotopic to $N(\phi_*(a))$.

If $\phi(N(a))$ is isotopic to $N(\phi_*(a))$, then we may choose OB cobordism Σ so that it has no hyperbolic singular points, hence $c(\phi, a) = 0$. We call such an OB cobordism *trivial*.

We observe the following properties of the function c .

Proposition 5.8. *Let $\phi, \psi \in \text{MCG}(S, \partial S)$ and $a, a' \in H_1(S, \partial S)$. We have:*

- (1) $c(\phi, a + a') = c(\phi, a) + c(\phi, a')$.
- (2) $c(\psi\phi, a) = c(\phi, a) + c(\psi, \phi_*(a))$.
- (3) *Let C be a simple closed curve which does not intersect the walls. Let T_C denote the right-handed Dehn twist along C . We have $c(T_C, a) = 0$ for any $a \in H_1(S, \partial S)$.*
- (4) *Let C be a simple closed curve in S such that $a \cdot [C] = 0$. Then $c(T_C, a) = 0$.*

In particular, (1) and (2) imply that the function c induces a crossed homomorphism

$$\mathcal{C} : \text{MCG}(S, \partial S) \rightarrow \text{Hom}(H_1(S, \partial S), \mathbb{Z}) \simeq H^1(S; \mathbb{Z}); \quad \phi \mapsto c(\phi, -).$$

Proof. First we prove (1). Let $\phi N(a) \xrightarrow{\Sigma} N(\phi_* a)$ and $\phi N(a') \xrightarrow{\Sigma'} N(\phi_* a')$ be OB cobordisms. We place the surfaces Σ and Σ' so that

- $\phi N(a)$ and $\phi N(a')$ in S_0 have the minimal geometric intersection, and
- $N(\phi_* a)$ and $N(\phi_* a')$ in S_1 have the minimal geometric intersection.

Since ϕ is a diffeomorphism fixing the boundary point-wise, $N(a)$ and $N(a')$ have the minimal geometric intersection.

Let $H \subset S$ be one of the once-punctured torus chambers. Then $H \cap N(a)$ and $H \cap N(a')$ are torus links. Suppose that $H \cap N(a)$ is the (p, q) -torus link and $H \cap N(a')$ is the (p', q') -torus link. Hence $H \cap N(a + a')$ is the $(p + p', q + q')$ -torus link. We resolve the double points of $H \cap (N(a) \cup N(a'))$ as shown in Figure 22, and call the resulting set of curves $A_{H, a, a'}$. Note that $[A_{H, a, a'}] = [H \cap N(a + a')]$ in $H_1(S, \partial S)$. We compare $A_{H, a, a'}$ and $H \cap N(a + a')$:

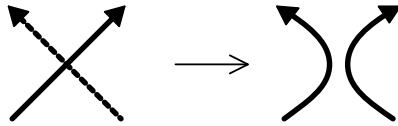


FIGURE 22. Smoothing intersections.

- (i) Suppose that $(\mathbf{sgn}(p), \mathbf{sgn}(q)) = (-\mathbf{sgn}(p'), -\mathbf{sgn}(q'))$. Let $n = \min\{|p|, |q|, |p'|, |q'|\}$. Then $A_{H,a,a'}$ is a disjoint union of $H \cap N(a+a')$, n many circles bounding concentric discs oriented counterclockwise, and n many those clockwise. Remove the circles as Figure 18, which yields n many negative and n many positive hyperbolic points. Hence we obtain an OB cobordism

$$H \cap N(a+a') \xrightarrow{\Sigma_{H,a,a'}} A_{H,a,a'}$$

with $d(\Sigma_{H,a,a'}) = n - n = 0$.

- (ii) Suppose that $(\mathbf{sgn}(p), \mathbf{sgn}(q)) \neq (-\mathbf{sgn}(p'), -\mathbf{sgn}(q'))$. In this case, we have $A_{H,a,a'} = H \cap N(a+a')$. Hence we obtain a trivial OB cobordism $H \cap N(a+a') \xrightarrow{\Sigma_{H,a,a'}} A_{H,a,a'}$ with $d(\Sigma_{H,a,a'}) = 0$.

Next let $H \subset S$ be the k -th annulus chamber. Recall the properly embedded path $\rho'_k \subset H$ joining the boundary circles γ_1 and γ_k (cf. Figure 15). We may suppose that $H \cap N(a) = n\rho'_k$ and $H \cap N(a') = n'\rho'_k$. Then $H \cap N(a+a') = (n+n')\rho'_k$.

- (iii) If $\mathbf{sgn}(n) = \mathbf{sgn}(n')$, let $A_{H,a,a'} := (H \cap N(a)) \sqcup (H \cap N(a'))$. Then $A_{H,a,a'} = H \cap N(a+a')$. Again we obtain a trivial OB cobordism $H \cap N(a+a') \xrightarrow{\Sigma_{H,a,a'}} A_{H,a,a'}$ with $d(\Sigma_{H,a,a'}) = 0$.
- (iv) If $\mathbf{sgn}(n) \neq \mathbf{sgn}(n')$, join $N(a)$ and $N(a')$ by description arcs from the innermost pairs of ρ'_k and $-\rho'_k$ to introduce $m = \min\{|n|, |n'|\}$ many hyperbolic singularities of the same sign $= \varepsilon$. See Figure 23. Call the resulting set of curves $A_{H,a,a'}$. Then $A_{H,a,a'}$ is the disjoint union of $H \cap N(a+a')$ and null-homologous nested arcs. This yields an OB cobordism $H \cap N(a+a') \xrightarrow{\Sigma_{H,a,a'}} A_{H,a,a'}$ with $d(\Sigma_{H,a,a'}) = \varepsilon m$.

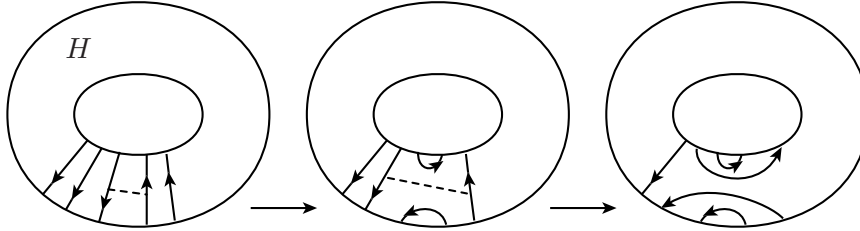


FIGURE 23. Case (iv). (Left) Curves $N(a) \sqcup N(a')$. (Right) $A_{H,a,a'}$.

Let

$$A_0 = \phi\left(\bigsqcup_{H \subset S} A_{H,a,a'}\right), \quad \Sigma_0 = (\phi \times \text{id})\left(\bigsqcup_{H \subset S} \Sigma_{H,a,a'}\right),$$

where the disjoint unions are taken for all the $g+r-1$ many chambers H in S . Now we obtain an OB cobordism

$$\phi N(a+a') \xrightarrow{\Sigma_0} A_0 \quad \text{with} \quad d(\Sigma_0) = \sum_H d(\Sigma_{H,a,a'}).$$

We repeat the arguments parallel to (i)–(iv) by replacing a by ϕ_*a and a' by ϕ_*a' . Namely, for each chamber H we construct a set of curves $A_{H,\phi_*a,\phi_*a'}$ from $H \cap (N(\phi_*a) \cup N(\phi_*a'))$ and an OB cobordism

$$A_{H,\phi_*a,\phi_*a'} \xrightarrow{\Sigma_{H,\phi_*a,\phi_*a'}} H \cap N(\phi_*(a + a')).$$

Let

$$A_1 = \bigsqcup_{H \subset S} A_{H,\phi_*a,\phi_*a'}, \quad \Sigma_1 = \bigsqcup_{H \subset S} \Sigma_{H,\phi_*a,\phi_*a'},$$

then we obtain an OB cobordism

$$A_1 \xrightarrow{\Sigma_1} N(\phi_*(a + a')) \quad \text{with} \quad d(\Sigma_1) = \sum_H d(\Sigma_{H,\phi_*a,\phi_*a'}).$$

Claim 5.9. *We have $d(\Sigma_1) = -d(\Sigma_0)$.*

Proof. For cases (i, ii, iii), we have $d(\Sigma_{H,\phi_*a,\phi_*a'}) = 0$. For case (iv), i.e., H is the k -th annulus chamber, since $\phi = \text{id}$ near ∂S , we have $H \cap N(\phi_*a) = H \cap N(a) = n\rho'_k$ and $H \cap N(\phi_*a') = H \cap N(a') = n'\rho'_k$. Therefore, the OB cobordism $\Sigma_{H,\phi_*a,\phi_*a'}$ is given by the reverse direction as depicted in Figure 23. Recalling that $d(\Sigma_{H,a,a'}) = \varepsilon m$, we have $d(\Sigma_{H,\phi_*a,\phi_*a'}) = -\varepsilon m$. This concludes the claim. \square

Next we construct an OB cobordism $A_0 \xrightarrow{\Sigma^+} A_1$. To this end, we recall that the surfaces Σ and Σ' are obtained by sequence of configuration changes (cf. Figure 17). In general, a description arc, δ , of a hyperbolic singularity for, say Σ , may intersect Σ' as shown in the top left sketch of Figure 24, where the black arc (resp. red arcs) belongs to Σ (resp. Σ') and the dashed arc indicates a description arc. By isotopy, we make δ, Σ, Σ' have no triple

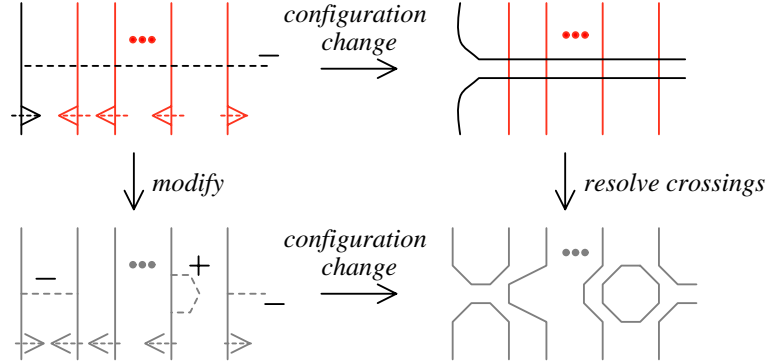


FIGURE 24. Modification of configuration changes.

intersection points and δ and Σ' attain the minimal geometric intersection. We project δ to the diagram of A_0 then split the projected δ into several description arcs for A_0 as in the left bottom sketch of Figure 24: Let p_0, \dots, p_k be the consecutive points of $\delta \cap A_0$. We

split δ into k arcs $\delta_1, \dots, \delta_k$ so that δ_i joins p_{i-1} and p_i . If the normals at p_{i-1} and p_i are pointing the same direction, then we discard δ_i . The set of surviving δ_i s replace the original δ and serve as description arcs for A_0 . If the sign of original δ is ε , then the algebraic count of the replacing δ_i s is also ε . This modification of configuration changes yields an OB cobordism $A_0 \xrightarrow{\Sigma^+} A_1$. By the construction of Σ^+ , we have $d(\Sigma^+) = d(\Sigma) + d(\Sigma')$.

Finally we obtain OB cobordisms

$$\phi N(a + a') \xrightarrow{\Sigma_0} A_0 \xrightarrow{\Sigma^+} A_1 \xrightarrow{\Sigma_1} N(\phi_*(a + a')).$$

By Claim 5.9,

$$c(\phi, a + a') = d(\Sigma_0) + d(\Sigma^+) + d(\Sigma_1) = d(\Sigma) + d(\Sigma') = c(\phi, a) + c(\phi, a').$$

We proceed to show (2). Let $\phi N(a) \xrightarrow{\Sigma} N(\phi_*(a))$ be an OB cobordism. Extend $\psi \in \text{MCG}(S, \partial S)$ to a diffeomorphism $\tilde{\psi} = \psi \times id : S \times [0, 1] \rightarrow S \times [0, 1]$ and we obtain an OB cobordism $\psi\phi(N(a)) \xrightarrow{\tilde{\psi}\Sigma} \psi(N(\phi_*(a)))$. Now let us take an OB cobordism $\psi(N(\phi_*(a))) \xrightarrow{\Theta} N(\psi_*\phi_*(a))$. Gluing $\tilde{\psi}\Sigma$ and Θ , we obtain an OB cobordism $\psi\phi(N(a)) \xrightarrow{\tilde{\psi}\Sigma} \xrightarrow{\Theta} N(\psi_*\phi_*(a))$. Since $\tilde{\psi}$ preserves the signs and the number of hyperbolic singularities, $d(\Sigma) = d(\tilde{\psi}\Sigma)$. This yields the desired equation.

To see (3), we observe that if a simple closed curve C does not intersect the walls, then $T_C(N(a))$ is in the normal form for any $a \in H_1(S, \partial S)$, i.e., $T_C(N(a)) = N(T_C a)$. Consider the product $\Sigma = T_C(N(a)) \times I$ which yields the trivial OB cobordism $T_C(N(a)) \xrightarrow{\Sigma} N(T_C a)$. Since the foliation is trivial, $c(T_C, a) = 0$.

Finally, we prove (4). We construct an OB cobordism $T_C N(a) \xrightarrow{\Sigma} N(a) = N(T_C a)$ with $d(T_C N(a), N(a)) = 0$ as follows. Since $[C] \cdot [T_C N(a)] = [C] \cdot a = 0$, by applying the configuration changes, described in Figure 17, to a portion of the multi-curve $T_C N(a)$ that lives in a small collar neighborhood of C , we can modify $T_C N(a)$ so that it is disjoint from C . For example, the left sketch in Figure 25 depicts the case when the geometric

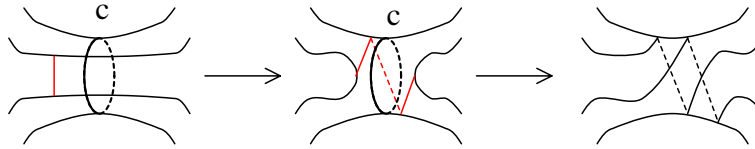


FIGURE 25. Untwisting multi-curve $T_C N(a)$ (left) to obtain $N(a)$ (right).

intersection number $i(T_C N(a), C) = 2$, where the red arcs indicate description arcs for hyperbolic singularities. Suppose that the sign of the hyperbolic singularity corresponding to the configuration change is ε . Next, we add a description arc of sign $-\varepsilon$ to the deformed $T_C N(a)$ (cf. the middle sketch) so that the corresponding configuration change yields the

multi-curve $N(a)$ (cf. the right sketch). This defines an OB cobordism $T_C N(a) \xrightarrow{\Sigma} N(a)$ which satisfies $d(T_C N(a), N(a)) = 0$.

When the geometric intersection number is greater than 2, a similar construction applies. Especially the sum of the total algebraic count of the signs in the first operation and the second operation is 0. \square

5.3. The function c : Planar surface case.

In this section, we study the function c for the case when $S = S_{0,r}$ a planar surface with r boundary components. We adopt the same notations in Section 5.1. The next proposition essentially has been proved in [37], by direct analysis of the OB cobordism (though this terminology is not explicitly used). Based on the fact that c is a crossed homomorphism, here we can give more detailed expression of c .

Recall the arcs ρ'_j and loops ρ_j ($j = 1, \dots, r-1$) specified in Figure 15. Under Poincaré duality $H_1(S, \partial S; \mathbb{Z}) \simeq H^1(S; \mathbb{Z})$; $[\rho'_j] \mapsto \text{PD}[\rho'_j]$, we may view $\{[\rho'_j]\}_{j=1}^{r-1}$ as a basis of $H^1(S)$. Let $\langle \cdot, \cdot \rangle$ denote the natural pairing of cohomology and homology. Then we have $\langle [\rho'_i], [\rho_j] \rangle = [\rho'_i] \cdot [\rho_j] = \delta_{i,j}$ the Kronecker delta.

Proposition 5.10. *Let $S = S_{0,r}$ be the planar surface with r boundary components. For $a \in H_1(S, \partial S; \mathbb{Z})$ the function c is characterized by the following:*

$$(5.5) \quad c(\phi, a) = \sum_{i=1}^{r-1} \langle [\rho'_i], \phi_* a - a \rangle - \sum_{j=1}^{r-1} \langle a, [\rho_j] \rangle \langle [\rho'_j], \phi_* [\rho'_j] - [\rho'_j] \rangle$$

where $\phi_* a - a$ and $\phi_* [\rho'_i] - [\rho'_i]$ are regarded as elements of $H_1(S; \mathbb{Z})$.

Moreover, let $\{t_{i,j}\}_{1 \leq i,j \leq r-1}$ be the matrix with $[\rho'_i] - \phi_* [\rho'_i] = \sum_{j=1}^{r-1} t_{i,j} [\rho_j]$ and suppose that $a = \sum_{j=1}^{r-1} x_j [\rho'_j]$. Then we can restate (5.5) as follows.

$$(5.6) \quad c(\phi, a) = - \sum_{j=1}^{r-1} x_j \sum_{1 \leq i \leq r-1, i \neq j} t_{j,i}$$

Remark 5.11. For planar case, Proposition 5.10 shows that our function c is completely determined by the homological data, namely, the map $\phi_* - id : H_1(S, \partial S) \rightarrow H_1(S)$.

Proof. For $j = 1, \dots, r-1$, we have

$$(5.7) \quad c(\phi, [\rho'_j]) = \sum_{1 \leq i \leq r-1, i \neq j} \langle [\rho'_i], \phi_* [\rho'_j] - [\rho'_j] \rangle$$

by the following reasons.

We recall that $c(\phi, [\rho'_j])$ counts algebraically the hyperbolic singularities produced by the configuration changes (cf. Figure 17) of the multi-curve $\phi(\rho'_j)$ where it crosses the walls. We write ϕ as a product of special type of Dehn twists, that are used in [37] and denoted by $A_{k,l}, A_m$ there. We observe that a \pm Dehn twist that involves the i -th and j -th bindings ($i \neq j$) contributes ± 1 hyperbolic singularity for the OB cobordism $\phi(\rho'_j) \xrightarrow{\Sigma} N(\phi_* [\rho'_j])$. But a Dehn twist around a single binding γ_k for any $k = 1, \dots, r-1$, does not contribute

any hyperbolic singularity to the OB cobordism. Since the quantity $\langle [\rho'_i], \phi_*[\rho'_j] - [\rho'_j] \rangle$ counts algebraically the number of circles in $N(\phi_*[\rho'_j])$ around the binding γ_i , equation (5.7) follows.

Recall that $\{[\rho_i] \in H_1(S)\}_{i=1}^{r-1}$ is the dual basis of $\{[\rho'_i] \in H_1(S, \partial S)\}_{i=1}^{r-1}$. Thus, we may express $a \in H_1(S, \partial S) \simeq H^1(S)$ as: $a = \sum_j \langle a, [\rho_j] \rangle [\rho'_j]$. By the crossed homomorphism property of the function c (Proposition 5.8), we can deduce (5.5) as follows:

$$\begin{aligned} c(\phi, a) &= \sum_j \langle a, [\rho_j] \rangle c(\phi, [\rho'_j]) \\ &\stackrel{(5.7)}{=} \sum_j \sum_{i \neq j} \langle [\rho'_i], \langle a, [\rho_j] \rangle (\phi_*[\rho'_j] - [\rho'_j]) \rangle \\ &= \sum_i \langle [\rho'_i], \phi_* a - a \rangle - \sum_j \langle a, [\rho_j] \rangle \langle [\rho'_j], \phi_*[\rho'_j] - [\rho'_j] \rangle. \end{aligned}$$

Now plugging the relation $[\rho'_j] - \phi_*[\rho'_j] = \sum_{i=1}^{r-1} t_{j,i} [\rho_i]$ to (5.7) we obtain

$$(5.8) \quad c(\phi, [\rho'_j]) = - \sum_{1 \leq i \leq r-1, i \neq j} t_{j,i}.$$

Linearly extending (5.8) to a general element $a = \sum_j x_j [\rho'_j]$ we obtain (5.6). \square

Remark 5.12. Since $\phi_*[\rho'_j] = [\rho'_j] \in H_1(S, \partial S)$, we have $c(\psi, \phi_*[\rho'_j]) = c(\psi, [\rho'_j])$. Therefore, when S is planar the property (2) in Proposition 5.8 can be restated as

$$c(\psi\phi, [\rho'_j]) = c(\phi, [\rho'_j]) + c(\psi, [\rho'_j]).$$

By using Theorem 5.4 and Proposition 5.10, now we can deduce the self-linking number formulae in [36], [37]. Let a_σ (resp. a_{ρ_j}) be the exponent sum of the the braid generators $\{\sigma_i\}_{i=1}^{n-1}$ (resp. ρ_j) in the braid word $b = b_1^{\varepsilon_1} b_2^{\varepsilon_2} \cdots b_l^{\varepsilon_l}$. Let $a = \sum_{i=1}^{r-1} s_i [\rho'_i] \in H_1(S, \partial S)$, the homology class introduced in the proof of Claim 5.2 such that $[b] = a - \phi_* a$.

Corollary 5.13 (The self-linking numbers for planar open books [37]). *With the notations above, the self-linking number is given by the following formula.*

$$sl(\hat{b}, [\Sigma]) = -n + a_\sigma + \sum_{j=2}^r a_{\rho_j} (1 - s_j) - \sum_{j=1}^{r-1} s_j \sum_{1 \leq i \leq r-1, i \neq j} t_{j,i}$$

Proof. Since $[b_i] \cdot [b_j] = 0$ for all $b_i, b_j \in \{\rho_1, \dots, \rho_{r-1}, \sigma_1, \dots, \sigma_{n-1}\}$, we have

$$\widehat{\exp}(b) = \sum_{i=1}^l \varepsilon_i = a_\sigma + \sum_{j=1}^{r-1} a_{\rho_j},$$

and since $[\rho'_j] \cdot [\rho_k] = \delta_{j,k}$ we have

$$\phi_*(a) \cdot [b] = (a - [b]) \cdot [b] = \left(\sum_{j=1}^{r-1} s_j [\rho'_j] \right) \cdot \left(\sum_{i=1}^l \varepsilon_i [b_i] \right) = \sum_{j=1}^{r-1} a_{\rho_j} s_j.$$

Hence by Theorem 5.4 and Proposition 5.10 we have:

$$\begin{aligned} sl(\hat{b}, \Sigma) &= -n + \widehat{\exp}(b) - \phi_*(a) \cdot [b] + c(\phi, a) \\ &= -n + a_\sigma + \sum_{j=1}^{r-1} a_{\rho_j} (1 - s_j) - \sum_{j=1}^{r-1} s_j \sum_{1 \leq i \leq r-1, i \neq j} t_{j,i}. \end{aligned}$$

□

5.4. The function c : Surface with one boundary component.

Let $S = S_{g,1}$ be a genus g surface with one boundary component. When $g = 1$, since there is no wall Proposition 5.8-(3) implies that $c(\phi, a) = 0$ for all $\phi \in \text{MCG}(S_{1,1})$ and $a \in H_1(S, \partial S)$. Henceforth in this section we restrict our attention to the case $g \geq 2$.

We observe in the following example that, unlike the planar case (cf. Remark 5.11), the function \mathcal{C} is no longer completely determined by homological data of ϕ . In fact we will see in Proposition 5.16 that \mathcal{C} carries even deeper information of ϕ .

Example 5.14. Let us take simple closed curves C, C' and ρ as in Figure 26. Since C

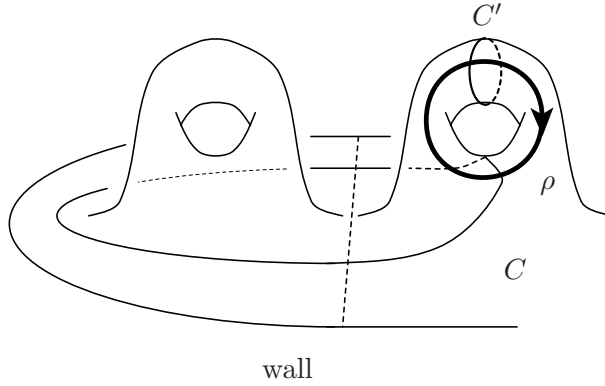
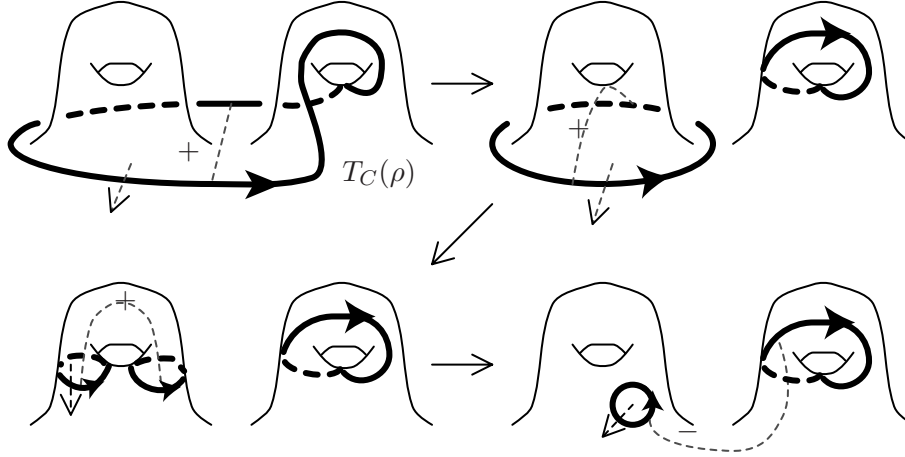


FIGURE 26. Mapping classes inducing the same action on homology, but taking different values for \mathcal{C} .

and C' bound a subsurface, T_C and $T_{C'}$ induce the same action on the homology groups $H_1(S; \mathbb{Z})$ and $H_1(S, \partial S; \mathbb{Z})$. As shown in Figure 27, we may modify the curve $T_C(\rho)$ into the normal form by introducing three positive hyperbolic singularities and one negative hyperbolic singularity. Hence $c(T_C, [\rho]) = 2$. On the other hand, C' does not intersect the walls, so $c(T_{C'}, [\rho]) = 0$.

In this section we use the following notations: Recall the circles $\rho_j, \rho'_j \subset S$ defined in Section 5.1. To distinguish elements of $H_1(S; \mathbb{Z})$ and $H_1(S, \partial S; \mathbb{Z}) \cong H^1(S; \mathbb{Z})$, we use the symbol $[\rho_j]$ ($j = 1, \dots, 2g$) to express the homology class of $H_1(S)$ represented by the circle ρ_j , and the symbol $[\rho'_j]$ for the relative homology class of $H_1(S, \partial S) \cong H^1(S)$ represented

FIGURE 27. Configuration change of $T_C(\rho)$ into the normal form.

by the circle ρ'_j . Note that since S has connected boundary, as a curve $\rho_j = \rho'_j$ for all j and as a group $H_1(S; \mathbb{Z}) \cong H_1(S, \partial S; \mathbb{Z}) \cong \mathbb{Z}^{2g}$.

Let $\langle \cdot, \cdot \rangle : H_1(S, \partial S; \mathbb{Z}) \times H_1(S; \mathbb{Z}) \rightarrow \mathbb{Z}$ denote the natural pairing of cohomology and homology, or the intersection pairing, i.e., $\langle \text{PD}[\rho'_j], [\rho_k] \rangle = [\rho'_j] \cdot [\rho_k]$. For simplicity, we denote $\text{PD}[\rho'_j]$ by $[\rho'_j]$ in the following. We have:

$$\langle [\rho'_j], [\rho_k] \rangle = \begin{cases} 1 & \text{if } (j, k) = (2i-1, 2i), \\ -1 & \text{if } (j, k) = (2i, 2i-1), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma := \pi_1(S)$, $\Gamma_1 := [\Gamma, \Gamma]$ the commutator subgroup, and $\Gamma_2 := [[\Gamma, \Gamma], \Gamma]$ the second term in the lower central series of Γ . Then the action of $\text{MCG}(S_{g,1})$ on Γ induces a representation

$$\varrho : \text{MCG}(S_{g,1}) \rightarrow \text{Aut}(\Gamma/\Gamma_2)$$

called the *(first) Johnson-Morita homomorphism*, cf. [42, p.199].

Let \mathcal{K} be the subgroup of $\text{MCG}(S_{g,1})$ generated by Dehn twists along separating simple closed curves in S . Johnson proved in [35] that for $g \geq 3$ we can identify $\mathcal{K} = \ker \varrho$. While, Proposition 5.8-(2)(4) imply that our crossed homomorphism \mathcal{C} also vanishes on \mathcal{K} . Hence it is natural to expect that the function c is governed by the image of ϱ , i.e., *metaabelian* information of the action of ϕ . In fact we will show that if $\varrho(\phi) = \varrho(\psi)$ then $\mathcal{C}(\phi) = \mathcal{C}(\psi)$.

This can be achieved by relating our function c to a well-known crossed homomorphism

$$k : \text{MCG}(S_{g,1}) \rightarrow H^1(S; \mathbb{Z})$$

which appears in various contexts in the theory of mapping class groups. (See Section 2 of [41] for concise overview.) Below we define k following Morita's [40, Sec 6] that is based on combinatorial group theory.

Definition 5.15. Let F_2 be the free group of rank two with generators α and β . Any element of F_2 is uniquely written in the form $\alpha^{\epsilon_1}\beta^{\delta_1}\cdots\alpha^{\epsilon_n}\beta^{\delta_n}$, where $\epsilon_i, \delta_i \in \{-1, 0, 1\}$. With this expression, we define a function $d : F_2 \rightarrow \mathbb{Z}$ by

$$d(\alpha^{\epsilon_1}\beta^{\delta_1}\cdots\alpha^{\epsilon_n}\beta^{\delta_n}) = \sum_{i=1}^n \delta_i \sum_{j=1}^i \epsilon_j.$$

Let α_i, β_i ($i = 1, \dots, g$) be generating curves of $\pi_1(S)$ as in Figure 28. Let $p_i : \pi_1(S) \rightarrow F_2$

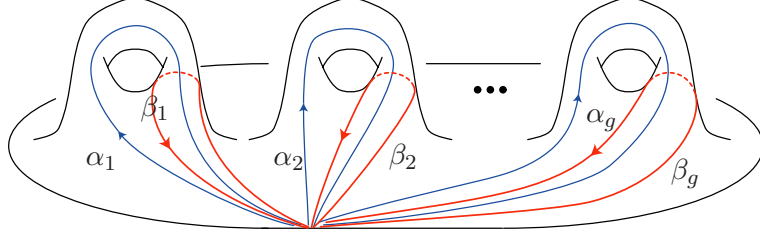


FIGURE 28.

be a homomorphism defined by

$$p_i(\gamma) = \begin{cases} \alpha & \text{if } \gamma = \alpha_i, \\ \beta & \text{if } \gamma = \beta_i, \\ 1 & \text{otherwise.} \end{cases}$$

Finally we define a map $k : \text{MCG}(S_{g,1}) \times H_1(S, \partial S) \rightarrow \mathbb{Z}$ by

$$k(\phi, a) = \sum_{i=1}^g d(p_i(\phi_*\gamma)) - d(p_i(\gamma))$$

where $\gamma \in \pi_1(S)$ represents $a \in H_1(S, \partial S)$. Morita proved in [40, Lemma 6.3] that $k(\phi, a)$ is a crossed homomorphism.

Now we give an explicit formula of the function c by using k . It provides a new geometric meaning of the classically known crossed homomorphism k : the signed count of the saddles in the OB cobordisms.

Proposition 5.16. *If $S = S_{g,1}$ has connected boundary and $g \geq 2$, then the function c is expressed as*

$$(5.9) \quad \begin{aligned} c(\phi, a) = & -2k(\phi, a) + \sum_{i=1}^g \langle [\rho'_{2i-1}] - [\rho'_{2i}], \phi_*a - a \rangle \\ & - \sum_{i=1}^g \langle a, [\rho_{2i-1}] \rangle \langle [\rho'_{2i}], \phi_*[\rho'_{2i}] - [\rho'_{2i}] \rangle \\ & - \sum_{i=1}^g \langle a, [\rho_{2i}] \rangle \langle [\rho'_{2i-1}], \phi_*[\rho'_{2i-1}] - [\rho'_{2i-1}] \rangle \end{aligned}$$

where $\phi_*a - a$ and $\phi_*[\rho'_j] - [\rho'_j]$ are regarded as elements of $H_1(S; \mathbb{Z})$.

Proof. Recall that the left hand side of (5.9) satisfies the crossed homomorphism properties (1), (2) in Proposition 5.8. Hence it is sufficient to verify (5.9) for some generating set of the mapping class group $\text{MCG}(S_{g,1})$.

We use the *Lickorish generators* of $\text{MCG}(S_{g,1})$. Let A_i, B_i ($i = 1, \dots, g$) and C_j ($j = 1, \dots, g-1$) be simple closed curves as shown in Figure 29. Lickorish proved that the Dehn

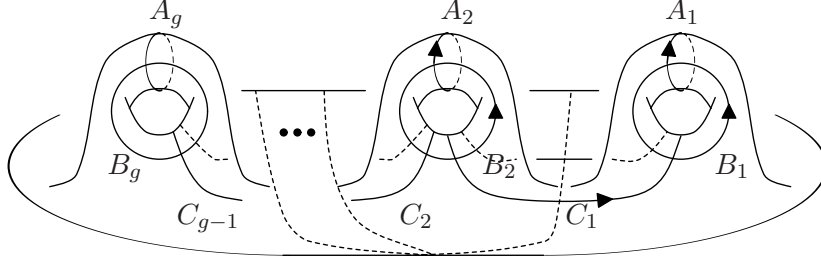


FIGURE 29. Generating curves for $\text{MCG}(S_{g,1})$.

twists along these $3g - 1$ curves generate $\text{MCG}(S_{g,1})$. With the orientations indicated in Figure 29, we have in $H_1(S; \mathbb{Z})$ that

$$[A_i] = [\rho_{2i-1}], \quad [B_i] = [\rho_{2i}], \quad \text{and} \quad [C_i] = -[\rho_{2i-1}] + [\rho_{2i+1}].$$

First of all, we observe that if $D \in \{A_i, B_i, C_i\}$ is disjoint from the loop ρ_j , then $c(T_D, [\rho'_j]) = k(T_D, [\rho'_j]) = 0$ and $T_{D*}[\rho'_j] - [\rho'_j] = 0$, thus the formula (5.9) holds. So we only need to consider the case when D has non-trivial intersection with ρ_j . There are four cases to study:

Case I: $(\phi, a) = (T_{A_i}, [\rho'_{2i}])$

Since A_i is disjoint from the walls, Proposition 5.8-(3) implies that $c(T_{A_i}, [\rho'_{2i}]) = 0$. On the other hand, $T_{A_i}(\rho_{2i}) = \rho_{2i-1}\rho_{2i} = \beta_{g-i}\alpha_{g-i}^{-1}$ in $\pi_1(S)$ hence $k(T_{A_i}, [\rho'_{2i}]) = d(\beta\alpha^{-1}) = 0$. Finally, observe that $T_{A_i*}[\rho'_{2i}] - [\rho'_{2i}] = [\rho_{2i-1}]$, hence

$$(\star) \quad := \sum_{k=1}^g \langle [\rho'_{2k-1}] - [\rho'_{2k}], \phi_*a - a \rangle = \langle -[\rho'_{2i}], [\rho_{2i-1}] \rangle = 1,$$

$$\begin{aligned} (\star\star) \quad &:= \sum_{k=1}^g \langle a, [\rho_{2k-1}] \rangle \langle [\rho'_{2k}], \phi_*[\rho'_{2k}] - [\rho'_{2k}] \rangle + \sum_{k=1}^g \langle a, [\rho_{2k}] \rangle \langle [\rho'_{2k-1}], \phi_*[\rho'_{2k-1}] - [\rho'_{2k-1}] \rangle \\ &= \langle [\rho'_{2i}], [\rho_{2i-1}] \rangle^2 = (-1)^2 = 1. \end{aligned}$$

Thus, the equality (5.9) holds.

Case II: $(\phi, a) = (T_{B_i}, [\rho'_{2i-1}])$

As in the Case I, B_i is disjoint from the walls, so $c(T_{B_i}, [\rho'_{2i-1}]) = 0$. On the other hand,

$T_{B_i}(\rho_{2i-1}) = \rho_{2i-1}\rho_{2i}^{-1} = \beta_{g-i}\alpha_{g-i}$, hence $k(T_{B_i}, [\rho'_{2i-1}]) = d(\beta\alpha) = 0$. Finally observe that $T_{B_i*}[\rho'_{2i-1}] - [\rho'_{2i-1}] = -[\rho_{2i}]$, hence

$$\begin{aligned} (\star) &= \langle [\rho'_{2i-1}], -[\rho_{2i}] \rangle = -1, \\ (\star\star) &= \langle [\rho'_{2i-1}], [\rho_{2i}] \rangle \langle [\rho'_{2i-1}], -[\rho_{2i}] \rangle = -1. \end{aligned}$$

Thus, the equality (5.9) holds.

Case III: $(\phi, a) = (T_{C_i}, [\rho'_{2i}])$

Observe that $c(T_{C_i}, [\rho'_{2i}]) = -1$. Since

$$T_{C_i}(\rho_{2i}) = \rho_{2i}\rho_{2i+1}^{-1}\rho_{2i}^{-1}\rho_{2i-1}\rho_{2i} = \alpha_{g-i}^{-1}\beta_{g-i-1}^{-1}\alpha_{g-i}\beta_{g-i}\alpha_{g-i}^{-1},$$

$k(T_{C_i}, [\rho'_{2i}]) = d(\beta\alpha^{-1}) + d(\beta^{-1}) = 0$. Finally, $T_{C_i*}[\rho'_{2i}] - [\rho'_{2i}] = [\rho_{2i-1}] - [\rho_{2i+1}]$, hence

$$\begin{aligned} (\star) &= 0, \\ (\star\star) &= \langle [\rho'_{2i}], [\rho_{2i-1}] \rangle \langle [\rho'_{2i}], -[\rho_{2i+1}] + [\rho_{2i-1}] \rangle = (-1)^2 = 1. \end{aligned}$$

Thus, the equality (5.9) holds.

Case IV: $(\phi, a) = (T_{C_i}, [\rho'_{2i+2}])$

In this case, $c(T_{C_i}, [\rho'_{2i+2}]) = 1$ and

$$T_{C_i}(\rho'_{2i+2}) = \rho_{2i}^{-1}\rho_{2i-1}^{-1}\rho_{2i}\rho_{2i+1}\rho_{2i+2} = \alpha_{g-i}\beta_{g-i}^{-1}\alpha_{g-i}^{-1}\beta_{g-i-1}\alpha_{g-i-1}^{-1}.$$

Hence $k(T_{C_i}, [\rho'_{2i+2}]) = d(\alpha\beta^{-1}\alpha^{-1}) + d(\beta\alpha^{-1}) = -1$. Finally, $T_{C_i*}[\rho'_{2i+2}] - [\rho'_{2i+2}] = -[\rho_{2i-1}] + [\rho_{2i+1}]$, hence

$$\begin{aligned} (\star) &= 0, \\ (\star\star) &= \langle [\rho'_{2i+2}], [\rho_{2i+1}] \rangle \langle [\rho'_{2i+2}], -[\rho_{2i-1}] + [\rho_{2i+1}] \rangle = (-1)^2 = 1. \end{aligned}$$

Thus, the equality (5.9) holds. These computations complete the proof. \square

As Morita states in [42, Remark 4.9] the crossed homomorphism k is obtained directly from the Johnson-Morita homomorphism $\varrho : \text{MCG}(S_{g,1}) \rightarrow \text{Aut}(\Gamma/\Gamma_2)$. He proves that the image of ϱ lies in $\frac{1}{2}\bigwedge^3 H$ (here H denotes $H_1(S; \mathbb{Z}) \cong H^1(S; \mathbb{Z})$), and that $k = C \circ \varrho$ where $C : \frac{1}{2}\bigwedge^3 H \rightarrow H$ is the contraction. Moreover, the cohomology class of k generates the cohomology group $H^1(\text{MCG}(S_{g,1}); H^1(S; \mathbb{Z})) \cong \mathbb{Z}$, [40, Proposition 6.4]. Thus it is natural that k appears in the description of our crossed homomorphism c .

Theorem 5.4 and Proposition 5.16 give a new relationship between the contact structures of 3-manifolds and the Johnson-Morita homomorphisms.

This observation develops into the following question:

Let Γ_i be the i -th term in the lower central series of $\Gamma = \pi_1(S)$. Assume that we have $a \in \Gamma/\Gamma_{i-1}$ such that $[b] = \phi_* a \cdot a^{-1}$ as an element of Γ/Γ_{i-1} . Our result roughly says that, in the case $i = 2$, if we choose a homology class $a \in \Gamma/\Gamma_{i-1} = \Gamma/\Gamma_1 = H_1(S; \mathbb{Z})$ (in geometric point of view, this choice corresponds to the choice of Seifert surface of the transverse link $L = \widehat{b}$), then the Johnson-Morita homomorphism $\varrho : \text{MCG}(S_{g,1}) \rightarrow \text{Aut}(\Gamma/\Gamma_2)$ gives the self-linking number, which is the simplest invariant of transverse links. Now we ask

whether a similar phenomenon occurs in higher settings: Namely, do higher Johnson-Morita homomorphisms $\varrho_i : \text{MCG}(S_{g,1}) \rightarrow \text{Aut}(\Gamma/\Gamma_i)$ for $i \geq 3$ and a choice of $a \in \Gamma/\Gamma_{i-1}$ provide new invariants of transverse links?

5.5. The function c : General surface case.

Finally we give a complete description of the function c for general surfaces $S = S_{g,r}$. We use the same convention as in Section 5.4, that is, $[\rho'_j]$ is an element of $H_1(S, \partial S; \mathbb{Z}) \cong H^1(S; \mathbb{Z})$ and $[\rho_j]$ is an element of $H_1(S; \mathbb{Z})$.

Let $S' = S_{g,1}$ be the surface obtained from $S = S_{g,r}$ by filling the boundaries $\gamma_1, \dots, \gamma_{r-1}$ by discs and $i : S \rightarrow S'$ the canonical inclusion. Let $\pi : \text{MCG}(S_{g,r}) \rightarrow \text{MCG}(S_{g,1})$ be the forgetful map. Let us consider the pull-back $\pi^*k : \text{MCG}(S_{g,r}) \times H_1(S, \partial S) \rightarrow \mathbb{Z}$ of the crossed homomorphism k defined by

$$\pi^*k : (\phi, a) \mapsto k(\pi(\phi), i_*(a)).$$

For $1 \leq i \leq r-1+2g$, let

$$[\varsigma'_j] = \begin{cases} [\rho'_j] & \text{if } j = 1, \dots, r-1, \\ -[\rho'_{j+1}] & \text{if } i = r, r+2, \dots, r-2+2g, \\ [\rho'_{j-1}] & \text{if } i = r+1, r+3, \dots, r-1+2g. \end{cases}$$

In particular, we have $\langle [\varsigma'_i], [\rho_j] \rangle = \delta_{i,j}$.

By combining Proposition 5.10 and Proposition 5.16 we get an explicit formula of the function c .

Theorem 5.17 (A formula of function c). *Let $S = S_{g,r}$ be the surface with genus g and r boundary components. The function $c : \text{MCG}(S_{g,r}) \times H_1(S, \partial S; \mathbb{Z}) \rightarrow \mathbb{Z}$ has the following expression:*

$$c(\phi, a) = -2(\pi^*k)(\phi, a) + \sum_{j=1}^{2g+r-1} \langle [\varsigma'_j], \phi_*a - a \rangle - \sum_{j=1}^{2g+r-1} \langle a, [\rho_j] \rangle \langle [\varsigma'_j], \phi_*[\varsigma'_j] - [\varsigma'_j] \rangle$$

where $\phi_*a - a$ and $\phi_*[\varsigma'_j] - [\varsigma'_j]$ are regarded as elements of $H_1(S; \mathbb{Z})$.

6. NEW PROOFS TO FUNDAMENTAL THEOREMS IN CONTACT GEOMETRY

In this section using the open book foliation techniques, we give new proofs to two fundamental results in contact geometry, Bennequin-Eliashberg inequality [17] and Honda-Kazez-Matić's theorem about the characterization of monodromies of open books supporting tight contact structures [30].

Recall that an *overtwisted disc* is an embedded disc whose boundary is the limit cycle in the characteristic foliation. Thus an overtwisted disc always has *Legendrian* boundary. In the framework of the open book foliation, we consider an overtwisted disc with transverse boundary:

Definition 6.1. Let D be an embedded oriented disc whose boundary is a positive transverse unknot K . We say D is a *transverse overtwisted disc* if $sl(K, [D]) = 1$ and the graph $G_{-, -}$ (cf. Definition 3.10) is a tree.

Proposition 6.2. *If (S, ϕ) contains a transverse overtwisted disc then the compatible contact 3-manifold (M, ξ) contains an overtwisted disc.*

Later, we prove the converse in Corollary 6.5, hence the existence of transverse overtwisted disc is equivalent to the existence of usual overtwisted disc.

Proof. By Theorem 4.1 and applying the Giroux's elimination lemma (see [24, p187]) we can convert a transverse overtwisted disc into an overtwisted disc. \square

6.1. The Bennequin-Eliashberg inequality.

The following proof of the Bennequin-Eliashberg inequality is similar to Eliashberg's original proof which uses the characteristic foliation, but ours is simpler and more combinatorial.

Theorem 6.3 (Bennequin-Eliashberg inequality [17]). *If a contact 3-manifold (M, ξ) is tight, then for any null-homologous transverse link L and its Seifert surface Σ , the following inequality holds:*

$$sl(L, [\Sigma]) \leq -\chi(\Sigma)$$

The following is a key lemma to prove Theorem 6.3.

Lemma 6.4. *Let L be a null-homologous transverse link in a contact 3-manifold (M, ξ) and Σ be a Seifert surface for L . Assume that*

$$sl(L, [\Sigma]) > -\chi(\Sigma),$$

that is, Bennequin-Eliashberg inequality is violated. Then the graph $G_{-, -}$ contains a contractible component with no fake vertices.

Proof. Using Propositions 3.8 and 5.1, we assume that $sl(L, [\Sigma]) + \chi(\Sigma) = 2(e_- - h_-) > 0$, i.e., $e_- - h_- > 0$.

Let $\Gamma_1, \dots, \Gamma_k$ denote the connected components of the graph $G_{-, -}$. Let $f(\Gamma_i)$ be the number of the fake vertices of Γ_i and $e_-(\Gamma_i)$ the number of the negative elliptic points in Γ_i . Let $h_-(\Gamma_i)$ be the number of the edges in Γ_i . By Theorem 3.5, we may assume that $\mathcal{F}_{ob}(\Sigma)$ has no c -circles, hence the region decomposition (cf. Proposition 3.9) does not contain type ac , bc or cc regions, so $e_- = \sum_{i=1}^k e_-(\Gamma_i)$ and $h_- = \sum_{i=1}^k h_-(\Gamma_i)$. Since Γ_i is connected, the Euler characteristic of Γ_i satisfies that:

$$\chi(\Gamma_i) = (f(\Gamma_i) + e_-(\Gamma_i)) - h_-(\Gamma_i) \leq 1,$$

i.e., $e_-(\Gamma_i) - h_-(\Gamma_i) \leq 1 - f(\Gamma_i)$. Therefore we obtain that $e_-(\Gamma_i) - h_-(\Gamma_i) = 1$ if and only if $f(\Gamma_i) = 0$ and Γ_i is contractible. Now we have:

$$0 < e_- - h_- = \sum_{i=1}^k e_-(\Gamma_i) - \sum_{i=1}^k h_-(\Gamma_i) = \sum_{i=1}^k (e_-(\Gamma_i) - h_-(\Gamma_i)) \leq \sum_{i=1}^k (1 - f(\Gamma_i)).$$

Thus for some i , the equality $e_-(\Gamma_i) - h_-(\Gamma_i) = 1$ must hold, which implies that Γ_i is contractible and has no fake vertices. \square

Now we are ready to prove Theorem 6.3.

Proof of Theorem 6.3. Suppose that there exists a null-homologous transverse link L in (M, ξ) with Σ a Seifert surface such that $sl(L, [\Sigma]) > -\chi(\Sigma)$. We will show that ξ is overtwisted.

Fix an open book (S, ϕ) which supports ξ , and isotope L and Σ , with the transverse link class of L preserved, so that it admits an open book foliation $\mathcal{F}_{ob}(\Sigma)$. By Lemma 3.5, we may assume that $\mathcal{F}_{ob}(\Sigma)$ contains no c -circles, so the induced region decomposition (cf. Proposition 3.9) consists of aa -, ab -, and bb -tiles. By Lemma 6.4, the graph $G_{-, -}$ contains a contractible component Γ with no fake vertices.

Let \mathcal{R} be the set of regular b -arcs that end at the vertices of Γ . Clearly the closure $\overline{\mathcal{R}}$ contains Γ . Let $\mathcal{P} \subset \overline{\mathcal{R}}$ be the set of positive elliptic points in $\overline{\mathcal{R}}$ and the stable separatrices approaching to positive hyperbolic points in $\overline{\mathcal{R}}$. Since Γ is a tree with no fake vertices, $\overline{\mathcal{R}} \setminus \mathcal{P}$ is an open disc embedded in Σ .

If $\overline{\mathcal{R}}$ is not a disc (cf. Figure 30), then we can cut open $\overline{\mathcal{R}}$ along some stable separatrices

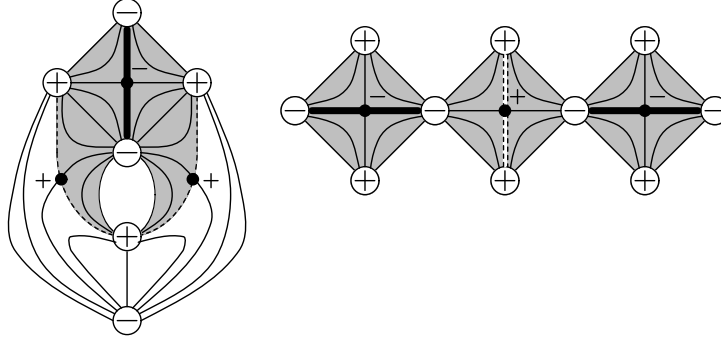


FIGURE 30. $\overline{\mathcal{R}}$ may not be a disc.

and/or elliptic points in \mathcal{P} to obtain a disc D whose interior $\text{Int}(D) = \overline{\mathcal{R}} \setminus \mathcal{P}$. Let $\mathcal{P}_\circ \subset \mathcal{P}$ be the set of such stable separatrices and elliptic points in \mathcal{P} needed to obtain D . Let $\lambda \in \mathcal{P}_\circ$ be a connected component, i.e., either a point (positive elliptic point) or an arc (a union of unstable separatrices joined at positive hyperbolic points and positive elliptic points).

In either case, we cut out $\overline{\mathcal{R}}$ along λ . Denote the copies of λ by λ_1, λ_2 . By isotopy that preserves the open book foliation $\mathcal{F}_{ob}(\overline{\mathcal{R}} \setminus \mathcal{P}_\circ)$, move a thin collar neighborhood $\nu(\lambda_1) \subset (\overline{\mathcal{R}} \setminus \lambda)$ of λ_1 so that $\lambda_1 \neq \lambda_2$. Figure 31 (resp. Figure 32) depicts the case when λ is a point (resp. the pair of stable separatrices for a single positive hyperbolic point).

We apply the cutting operation for all the connected components in \mathcal{P}_\circ and denote the resulting surface by D . By the construction, D is a disc and sitting on the surface Σ except a neighborhood of some part of the boundary ∂D . Note that ∂D consists of stable separatrices for positive hyperbolic points and positive elliptic points. We extend D by adding a collar neighborhood to ∂D so that the resulting surface, \tilde{D} , is a disc embedded in

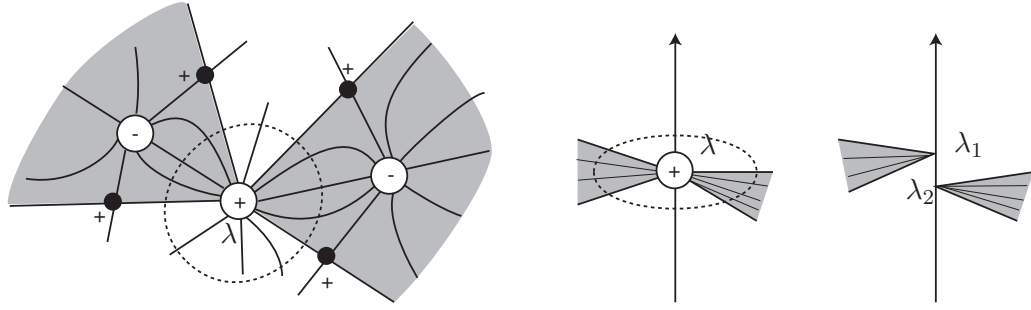


FIGURE 31. (Left) A neighborhood of a point λ in $\overline{\mathcal{R}}$ (the shaded region). (Right) Transition under the cutting operation.

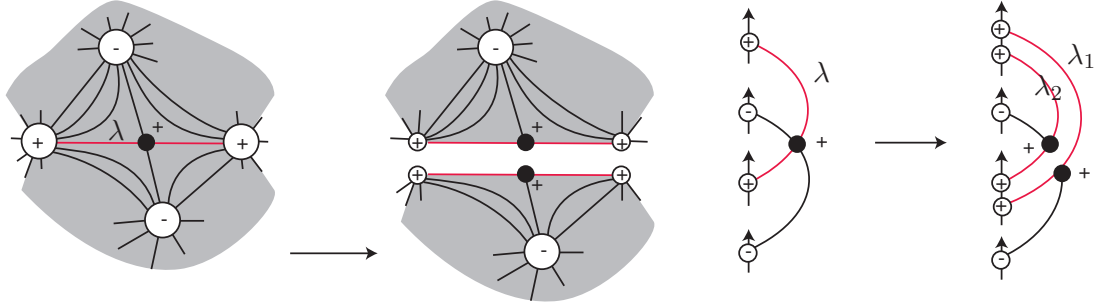


FIGURE 32. (Left) A neighborhood of an arc λ (red) in $\overline{\mathcal{R}}$. (Right) Transition on the page of the open book that contains λ under the cutting operation.

M , its boundary $\partial\tilde{D}$ is a positive transverse unknot, and the open book foliation $\mathcal{F}_{ob}(\tilde{D}\setminus D)$ on the collar has no singularities. See Figure 33. Hence \tilde{D} is a transverse overtwisted disc. By Proposition 6.2 we conclude that ξ is overtwisted. \square

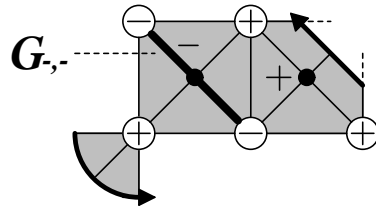


FIGURE 33. Construction of \tilde{D} .

Corollary 6.5. *If a contact 3-manifold $(M_{(S,\phi)}, \xi_{(S,\phi)})$ contains an overtwisted disc then (S, ϕ) contains a transverse overtwisted disc.*

Proof. Let $\Delta \subset (M, \xi)$ be an overtwisted disc. We orient Δ so that the elliptic point of $\mathcal{F}_\xi(D)$ has negative sign. Since Δ is embedded and the boundary $L = \partial\Delta$ is a Legendrian knot, [22, p.129] implies that we can take a collar neighborhood $\nu(\Delta)$ of Δ whose characteristic foliation $\mathcal{F}_\xi(\nu(\Delta))$ is sketched in Figure 34. Let $L^+ \subset \nu(\Delta)$ (dashed circle in

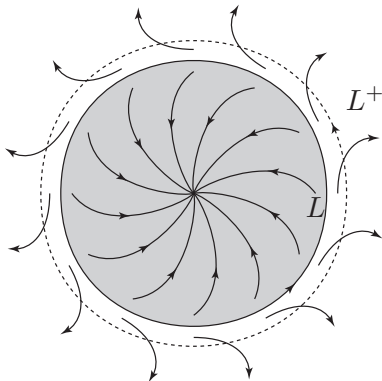


FIGURE 34. The characteristic foliation $\mathcal{F}_\xi(\nu(\Delta))$ and a positive transverse push off L^+ .

Figure 34) be a positive transverse push off of L . Let $\Delta^+ \subset \nu(\Delta)$ be the disc bounded by L^+ . Then $sl(L^+, [\Delta^+]) = 1$ and the Euler characteristic has $\chi(\Delta^+) = 1$. In particular, $sl(L^+, [\Delta^+]) > -\chi(\Delta^+)$. By the same argument as in the proof of Theorem 6.3, we can find a transverse overtwisted disc. \square

6.2. Right-veering monodromy and tightness.

In this section we give an alternative proof of Honda-Kazez-Matić's tightness criterion.

We first review the notion of right-veering mapping classes. See [30, Section 2] for precise definition. Let γ, γ' be isotopy classes (rel. to the endpoints) of oriented properly embedded arcs in S which start at the same point $*$ in ∂S . By abuse of notation, we use the same symbol γ for a representative of the isotopy class γ . We say that γ' lies *on the right side* of γ if there exist curves representing γ and γ' with the minimal geometric intersection such that around the starting point $*$, γ strictly lies on the right side of γ' . In such case, we denote $\gamma > \gamma'$.

Definition 6.6. [30, Definition 2.1] Let C be a boundary component of S . We say that $\phi \in \text{MCG}(S, \partial S)$ is *right-veering with respect to C* if $\gamma \geq \phi(\gamma)$ holds for any isotopy classes γ of properly embedded curves which start at a point on C . We say that ϕ is *right-veering* if ϕ is right-veering with respect to all the boundary components of S . In particular, the identity $\text{id} \in \text{MCG}(S, \partial S)$ is right-veering.

Theorem 6.7. [30, Theorem 1.1] *If ϕ is not right-veering, then $\xi_{(S,\phi)}$ is overtwisted.*

Remark 6.8. The converse of Theorem 6.7 also holds as we can find a proof in [30, p.444] where Eliashberg's classification of overtwisted contact structures [16] plays an important role. By Eliashberg's classification, an overtwisted contact structure admits an open book decomposition which is negative stabilization of some open book. Clearly such an open book has a non-right-veering monodromy.

Proof. If ϕ is not right-veering, then there exists a properly embedded oriented arc $\alpha \subset S$ such that $\phi(\alpha) \geq \alpha$. By [30, Lemma 5.2], there exists a sequence of properly embedded oriented arcs $\phi(\alpha) = \alpha_0 > \cdots > \alpha_k = \alpha$ such that $\alpha_0, \dots, \alpha_k$ have the same initial point, $*$, consecutive α_i and α_{i+1} have disjoint interiors and distinct terminal points p_i, p_{i+1} . In order to construct a transverse overtwisted disc, we orient α_i *against* the parametrization, i.e., the positive direction is from p_i to $*$. Let B_i be the binding to which p_i belongs. For positive tangent vectors $\vec{v}_\alpha \in T_{p_i}\alpha_i$ and $\vec{v}_B \in T_{p_i}B_i$ the ordered pair $\{\vec{v}_B, \vec{v}_\alpha\}$ yields a positive basis for $T_{p_i}S$.

Let $t_i = \frac{i}{k+1} \in [0, 1]$ where $i = 0, \dots, k+1$. Let β_i (resp. $\check{\beta}_i$) be a connected sub-arc of α_i whose endpoints are p_i and a point very close to p_i (resp. $*$). See the left sketch in Figure 35. Since $\phi = \text{id}$ near the bindings, $p_0 = p_k$ i.e., $\beta_0 = \beta_k$. To distinguish β_0 and

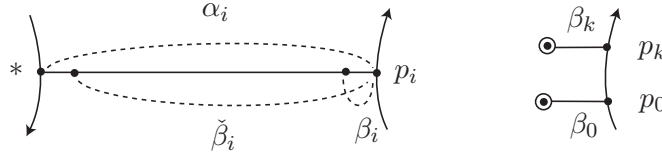


FIGURE 35. Arcs α_i, β_i , and $\check{\beta}_i$.

β_k we push p_k a little along the binding in the positive direction so that there is no other p_i sitting between p_0 and the pushed p_k . See the right sketch in Figure 35. Now the arcs β_0, \dots, β_k are pairwise disjoint. The orientation of α_i induces those of β_i and $\check{\beta}_i$.

We define sets of oriented arcs;

$$\begin{aligned} A_i &= \beta_0 \cup \cdots \cup \beta_k \text{ where } \beta_i \text{ is replaced by } \alpha_i, \text{ for } i = 0, \dots, k, \\ A_{k+1} &= \alpha \cup \beta_1 \cup \cdots \cup \beta_k. \end{aligned}$$

In the product domain $S \times [t_i, t_{i+1}]$ where $i = 0, \dots, k$, we construct a smooth surface D_i with $D_i \cap S_{t_i} = -A_i$ and $D_i \cap S_{t_{i+1}} = A_{i+1}$ in the following way:

As t increases from t_i to t_{i+1} ($i = 0, \dots, k-1$):

- For $j \neq i, i+1$, the arc β_j stays the same.
- The arc $\beta_{i+1} (\subset A_i \subset S_{t_i})$ smoothly extends to $\check{\beta}_{i+1}$ along α_{i+1} with the endpoint p_{i+1} fixed.
- Since $\alpha_i > \alpha_{i+1}$, in a small neighborhood N of $*$, the arc $\check{\beta}_{i+1}$ is on the right of α_i . So we can find an arc $\subset N$ (dashed in Figure 36) on the right side of α_i joining

α_i and $\check{\beta}_{i+1}$ so that the sign of the corresponding hyperbolic point h_i is positive. Passing the saddle point, they apart into arcs $\check{\beta}_i$ and α_{i+1} . See the passage from the left to right in Figure 36.

- The arc $\check{\beta}_i$ smoothly contracts to β_i along α_i with the endpoint p_i fixed. Hence we obtain A_{i+1} .

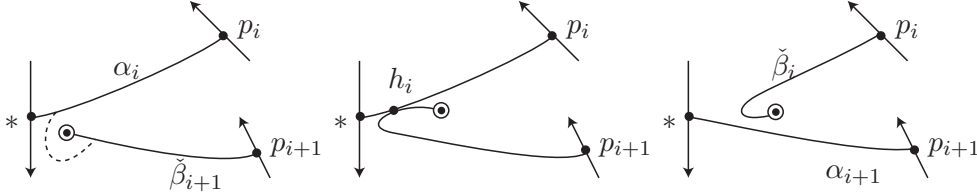


FIGURE 36. The construction of surface D_i .

The trace of \odot is in braid position in $S \times [t_i, t_{i+1}]$.

Similarly when $i = k$ as t increases from t_k to $t_{k+1} = 1$, arcs α_k and β_0 in A_k form a positive tangency h_k then apart into α and β_k in A_{k+1} . Rest of the β -arcs stay the same.

Since α_i and α_{i+1} do not intersect except at the initial point $*$, the surface D_i has no self-intersection.

Now we glue D_i and D_{i+1} along $A_{i+1} \subset S_{t_{i+1}}$ ($i = 0, \dots, k-1$) and obtain a surface $D_0 \cup \dots \cup D_k \subset S \times [0, 1]$ whose boundary is $-A_0 \cup A_{k+1}$. Since arcs β_1, \dots, β_k are very close to the bindings where the monodromy ϕ is the identity, we can identify A_0 and A_{k+1} under ϕ . The resulting surface in $M_{(S, \phi)}$ is denoted by D . The topological type of D is the 2-disc and its open book foliation $\mathcal{F}_{ob}(D)$ is depicted in Figure 37. In D , the point $*$

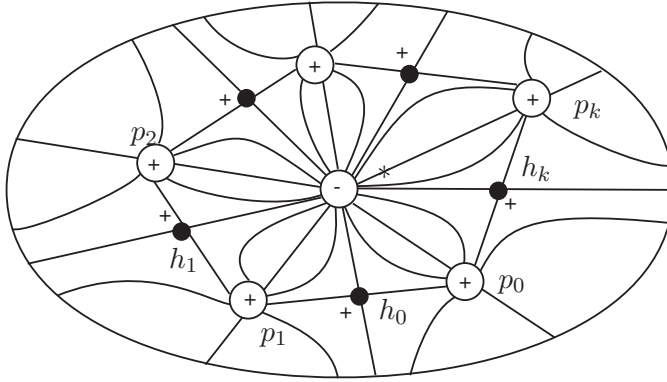


FIGURE 37. The transverse overtwisted disc D .

is a negative elliptic point, p_i ($i = 0, \dots, k$) is a positive elliptic point and h_i is a positive

hyperbolic point whose stable separatrices join p_i and p_{i+1} and unstable separatrices join $*$ and a point of ∂D . Clearly our D is a transverse overtwisted disc. \square

Remark 6.9. Reader may notice that our proof is similar to the original proof in [30], since both proofs are based on the same combinatorial lemma, [30, Lemma 5.2]. The crucial difference is that, in [30] this lemma is used to show the *existence* of bypass, the half of an overtwisted disc, whereas we use this lemma to *construct* a chain of positive elliptic and hyperbolic points surrounding the center negative elliptic point of an overtwisted disc. Our proof visualizes an overtwisted disc, in contrast to the original proof.

6.3. Right-veering monodromy and overtwistedness.

Theorem 6.7 states that if an open book (S, ϕ) supports a tight contact structure, then ϕ is right-veering. The converse does not hold in general. Indeed Honda-Kazez-Matić [30] show that for *every* contact structure there is a supporting open book with a right-veering monodromy. They utilize a sequence of stabilization to construct such an open book.

However in special cases, the converse holds: If $\chi(S) \geq -1$, then the contact manifold $(M_{(S,\phi)}, \xi_{(S,\phi)})$ is tight if and only if ϕ is right-veering. This fact is due to Eliashberg [17], Etnyre-Honda [19] and Goodman [28] when S is an annulus or a pair of pants, and Honda-Kazez-Matić [32, Theorem 1.2] when S is a once-punctured torus. We will study the problem when a right-veering open book supports a tight contact structure in our subsequent paper [33].

In [32, Question 6.2] Honda-Kazez-Matić ask whether a right-veering and non-destabilizable open book always supports a tight contact structure. Recently, Lekili [38] and Lisca [39] negatively answer to the question by constructing examples. They study open book decompositions of 3-manifolds whose tight contact structures are well-studied and classified (in [38] Poincaré homology 3-spheres, and in [39] lens spaces). In both constructions, the most technical points are showing that their open books indeed support overtwisted contact structures. Advanced tools such as Ozsváth-Szabo's Heegaard Floer invariants and properties of planar open books enable them to overcome the difficulty.

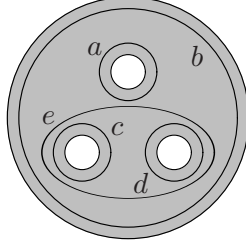
We generalize Lekili and Lisca's examples in Theorem 6.10 below. Our proof of overtwistedness is direct and does not require any knowledge of classification of tight contact structures of ambient manifolds or Ozsváth-Szabo's invariants.

Theorem 6.10. *Let S be a 2-sphere with four holes. Let a, b, c, d, e be simple closed curves on S as shown in Figure 38. For $h, i, k \geq 1$, let $\Phi_{h,i,k} = T_a^h T_b^i T_c T_d T_e^{-k-1}$ where T_x ($x = a, b, c, d, e$) denotes the right-handed Dehn twist along x . Then $\Phi_{h,i,k}$ is right-veering, and the open book $(S, \Phi_{h,i,k})$ is non-destabilizable and supports an overtwisted contact structure.*

Remark 6.11. Lekili's examples [38, Theorem 1.2 and Remark 4.1] are $\Phi_{2,i,1}$ ($i \leq 5$), and Lisca's examples [39, Theorem 1.1] are $\Phi_{h,1,l}$ ($h, l > 0$).

Proof. We owe [38] [39] the proof that $\Phi_{h,i,k}$ is right-veering and non-destabilizable. Hence we only show that $(S, \Phi_{h,i,k})$ supports an overtwisted contact structure.

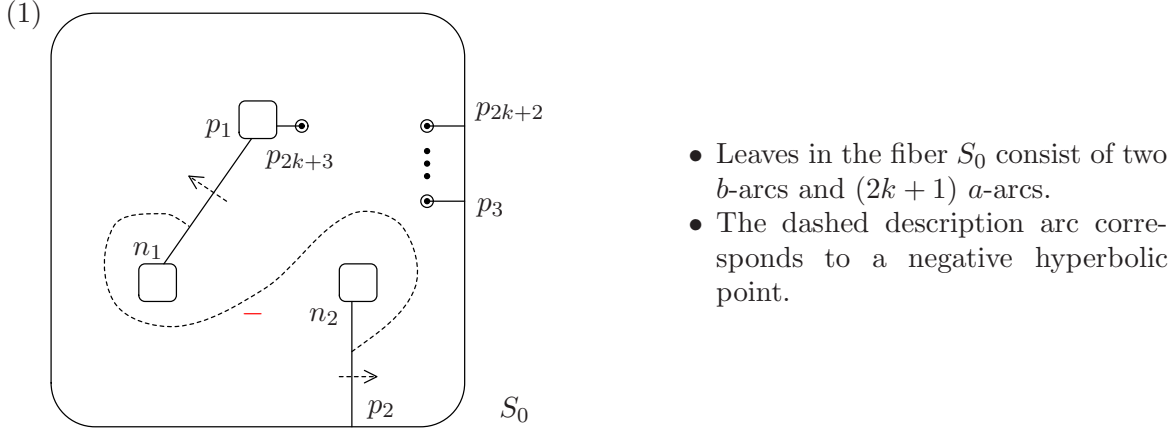
In Figure 39, we explicitly describe a transverse overtwisted disc D in the open book $(S, \Phi_{h,i,k})$. It contains two negative elliptic points n_1, n_2 and $(2k + 3)$ positive elliptic

FIGURE 38. Surface S .

points p_1, \dots, p_{2k+3} . We illustrate the intersection of D with six pages of the open book, including $D \cap S_0$ in Sketch (1), and $D \cap S_1$ in Sketch (7). Note that $\Phi_{h,i,k}(D \cap S_1) = D \cap S_0$. The open book foliation of D is depicted in Figure 40. Finally, we can verify that:

$$sl(\partial D, [D]) = -(e_+ - e_-) + (h_+ - h_-) = -((2k+3) - 2) + ((2k+3) - 1) = +1.$$

□



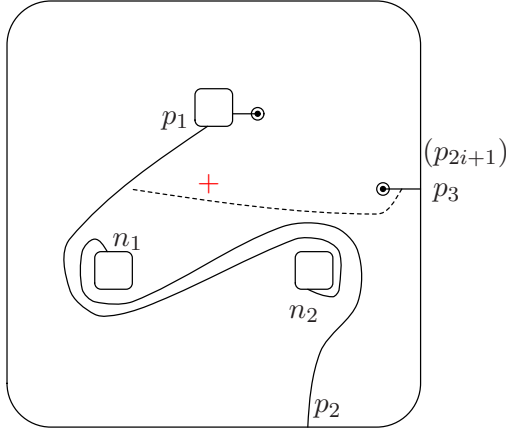
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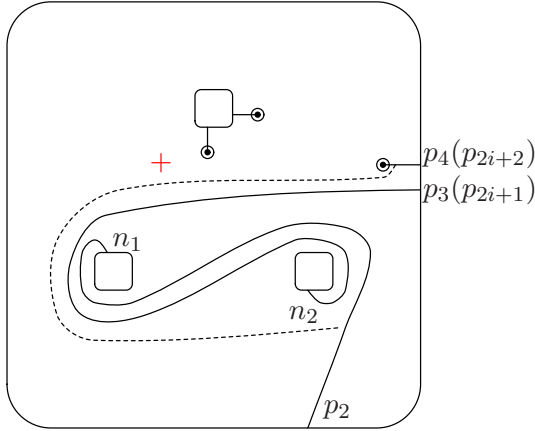
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(2)



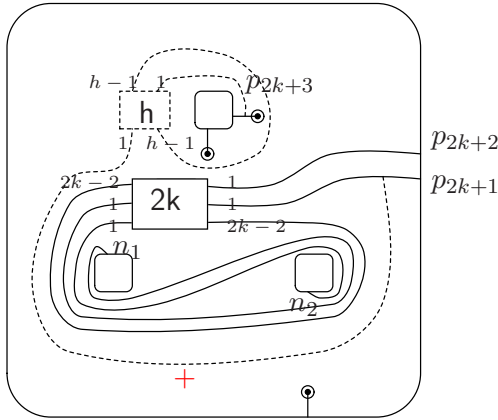
- In Sketches (2), ..., (7), we omit the a -arcs emanating from p_3, \dots, p_{2k+2} that are not involved in producing hyperbolic singular points.
- The description arc joining the b -arc emanating from n_2 and the a -arc from p_3 (or p_{2i+1} in the i -th iteration) represents a positive hyperbolic point.

(3)

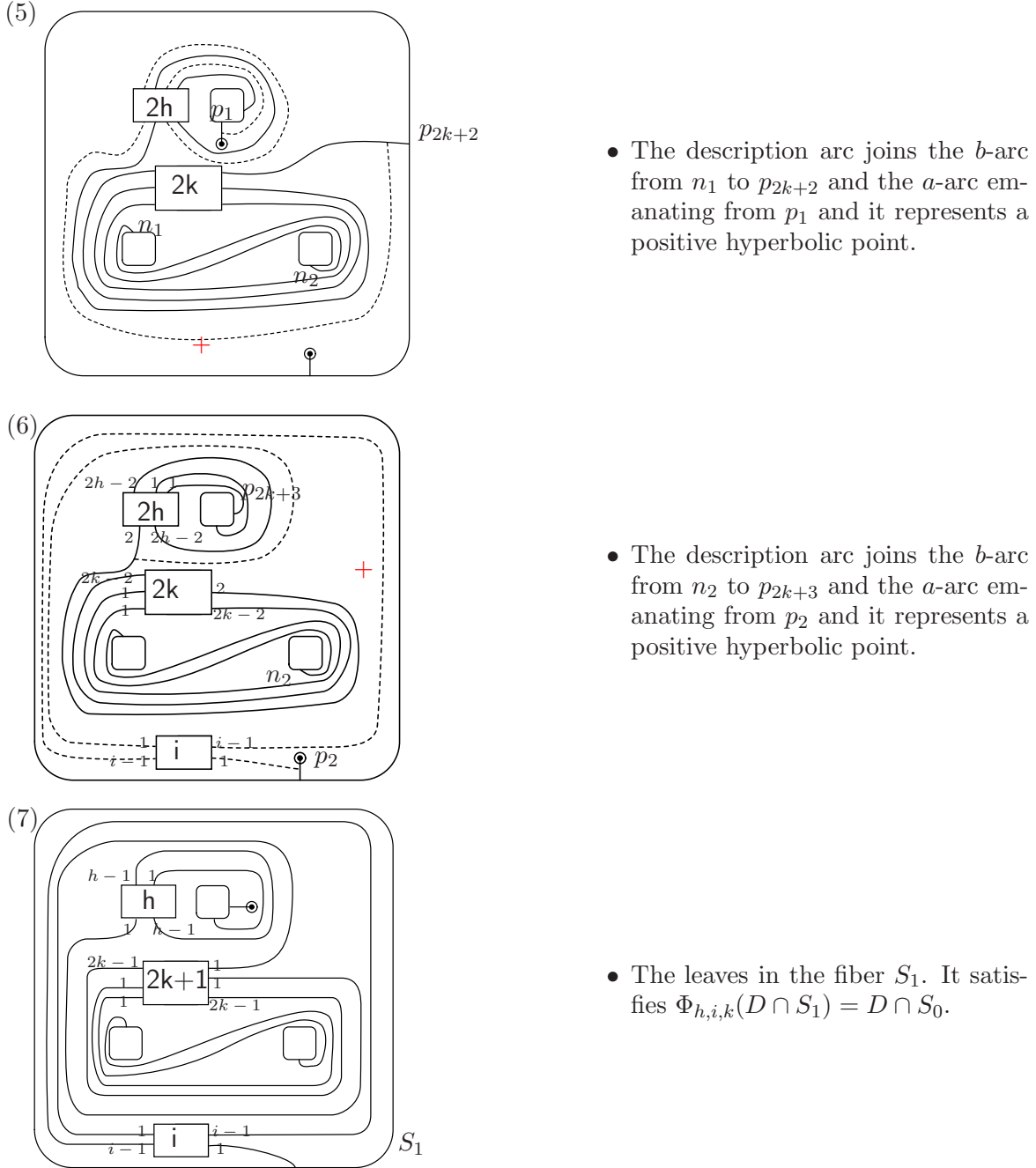


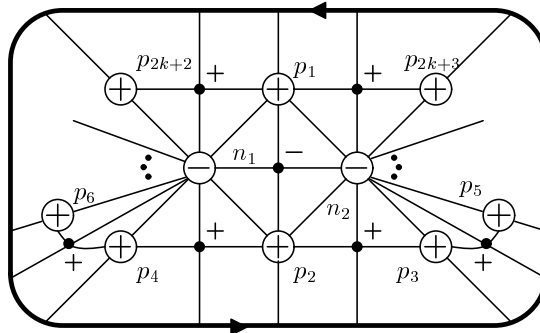
- The description arc joining the b -arc emanating from n_1 and the a -arc from p_4 (or, p_{2i+2} in the i -th iteration) represents a positive hyperbolic point.
- Iterate the Steps (2) and (3) for k times.

(4)



- The boxes labeled h , $2k$ contain parallel h , $2k$ arcs and edges out of the boxes are weighted as indicated.
- The description arc joins the b -arc from n_2 to p_{2k+1} and the a -arc emanating from p_{2k+3} and it represents a positive hyperbolic point.

FIGURE 39. Transverse overtwisted disc in the open book $(S, \Phi_{h,i,k})$.

FIGURE 40. The open book foliation $\mathcal{F}_{ob}(D)$.

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